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Existence of the universal attractor for the 3-D viscous primitive equations of large-scale moist atmosphere

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ABSTRACT

In this paper, we consider the initial–boundary value problem for the three-dimensional viscous primitive equations of large-scale moist atmosphere which are used to describe the turbulent behavior of long-term weather prediction and climate changes. By obtaining the existence and uniqueness of global strong solutions for the problem and studying the long-time behavior of strong solutions, we prove the existence of the universal attractor for the dynamical system generated by the primitive equations of large-scale moist atmosphere.

1. Introduction

In order to understand the mechanism of long-term weather prediction and climate changes, one can study the mathematical equations and models governing the motion of the atmosphere as the atmosphere is a specific compressible fluid (see, e.g., [19,29]). V. Bjerkness, one of the pioneers of meteorology, said that the weather forecasting can be considered as an initial boundary value problem in mathematical physics. In 1922, Richardson initially introduced the so-called primitive atmospheric equations which consisted of the hydrodynamic, thermodynamic equations with Coriolis force, cf. [30]. At that time, the primitive atmospheric equations were too complicated to be studied theoretically or to be solved numerically. To overcome this difficulty, some simple numerical models were introduced, such as the barotropic model formulated by Neumann etc. in [5] and the quasi-geostrophic model

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introduced by Charney and Philips in [8]. The 2-D and 3-D quasi-geostrophic models have been the subject of analytical mathematical study, cf., e.g., [2,3,6,7,13,14,28,37–39] and references therein.

Due to the considerable improvement in computer capacity and the development of atmospheric science, some mathematicians began to consider the primitive equations of atmosphere in the past two decades (see, e.g., [25–27,34] and references therein). In [25], by introducing viscosity terms and using some technical treatment, Lions, Temam and Wang obtained a new formulation of the primitive equations of large-scale dry atmosphere which was amenable to mathematical study. In a p-coordinate system, the new formulation of the primitive equations is a little similar to Navier–Stokes equations of incompressible fluid. By the methods used to solve Navier–Stokes system in [23], they obtained the existence of global weak solutions of the initial boundary value problem for the new formulation of the primitive equations. Moreover, under the assumptions that there exists a unique global strong solution for the problem with vertical viscosity and that H^1-norm of the strong solution is bounded uniformly in t, they established some physically relevant estimates for the Hausdorff and fractal dimensions of the attractor of the primitive equations with vertical viscosity. Without those assumptions, by the Trajectory Attractors Theory due to Vishik and Chepyzhov (cf. [12,35]), we obtained the existence of trajectory attractors for the large-scale moist atmospheric primitive equations in [16]. By taking advantage of the geostrophic balance and other geophysical consideration, several intermediate models have been the subject of studying the long-time dynamics and universal attractors in order to describe the mechanism of long-term weather prediction and climate dynamics (see, e.g., [9,10,21,36,37] and references therein).

In recent years, there were some mathematicians who considered the existence of strong solutions for the three-dimensional viscous primitive equations of large-scale atmosphere and ocean (see, e.g., [4,9–11,18,20,33,34] and references therein). In [18], Guillén-González et al. obtained the global existence of strong solutions to the primitive equations of large-scale ocean by assuming that the initial data are small enough, and also proved the local existence of strong solutions to the equation for all initial data. In [34], Temam and Ziane considered the local existence of strong solutions for the primitive equations of the atmosphere, the ocean and the coupled atmosphere–ocean. The papers [9–11] are devoted to considering the non-dimensional Boussinesq equations or modified models (see, e.g., [29,33]). In [9], Cao and Titi considered global well-posedness and finite-dimensional global attractor to a 3-D planetary geostrophic model. The paper [11] is devoted to studying the global well-posedness for the three-dimensional viscous primitive equations of large-scale ocean. In [11], Cao and Titi developed a beautiful approach, by which they obtained the fact that L^6-norm of the fluctuation $\tilde{v}$ of horizontal velocity is bounded uniformly in t. The estimate about L^6-norm of the fluctuation $\tilde{v}$ is a key proof in [11]. On the basis of the results of [11], we obtain the existence of (weak) universal attractors for the 3-D viscous primitive equations of the large-scale ocean in [17].

In the present paper we are interested in considering the existence of the universal attractor for the dynamical system generated by the primitive equations of large-scale moist atmosphere. In order to do that, we must study the existence, uniqueness and long-time behavior of global strong solutions to the initial–boundary value problem of the new formulation of large-scale moist atmospheric primitive equations (the problem is denoted by (IBVP) and will be given in Section 2). Our results are Proposition 3.1, Proposition 3.2, Proposition 3.3 and Theorem 3.4, where Theorem 3.4 is main. First, we obtain the global well-posedness of the problem (IBVP). Second, by studying the long-time behavior of the strong solution, we prove that H^1-norm of the strong solution is bounded uniformly in t, and also prove that the corresponding semigroup $\{S(t)\}_{t\geqslant 0}$ possesses a bounded absorbing set B_ρ in V (the definition of the space V will be given in Section 4.1), by which we construct a (weak) universal attractor $\mathcal{A}$. Here, the result about the universal attractors in this paper is more stronger than that in [16]. Since the global well-posedness of the 3-D incompressible Navier–Stokes system is still open, by Proposition 3.1, Proposition 3.2 Proposition 3.3 and Theorem 3.4, we prove rigorously in mathematics that the new formulation of large-scale moist atmospheric primitive equations is simpler than the incompressible Navier–Stokes system, which is consistent with the physical point of view.

Inspired by the methods used in [11], we prove the global well-posedness of (IBVP). However, there are three differences between our paper and [11]. First, our main aim is to study the long-time dynamics and the existence of the (weak) universal attractors for the large-scale moist atmospheric primitive equations, while the authors of [11] studied the global well-posedness of their considered model and did not consider the long-time behavior of strong solutions. By acquiring the uniform, in t, boundedness of L^3- and L^4-norm of the temperature T and the fluctuation $\tilde{v}$ of horizontal velocity v, which are most important in the proof of our main results (see Remark 5.5 for more discussion about the technique of estimates), we obtain the existence of the universal attractor for our considered dynamical system. Second, our ways to prove the global well-posedness are a little different from those in [11]. We prove the global well-posedness for our considered model by acquiring several a priori estimates about L^4-norm of T and $\tilde{v}$. Third, the new formulation of the large-scale moist atmospheric equations is more complicated than the model studied in [11]. If we let $a=0$, the model considered in this paper is similar to that in [11].

The paper is organized as follows. In Section 2, we pose the primitive equations of large-scale moist atmosphere. Main results of this paper are formulated in Section 3. In Section 4, we give our working spaces and some preliminaries. In Section 5, we make a priori estimates about the local strong solution and prove that H^1-norm of local strong solution is bounded uniformly in t, which is the kernel of the proof of our results. We prove main results of our paper in Sections 6, 7.

2. The three-dimensional viscous primitive equations of large-scale moist atmosphere

The three-dimensional viscous primitive equations of large-scale moist atmosphere in the pressure coordinate system (for details, we refer the reader to [16,22,25,26] and references therein) is written as

$$\frac{\partial v}{\partial t} + \nabla_v v + \omega \frac{\partial v}{\partial \xi} + \frac{f}{R_0} k \times v + \operatorname{grad} \Phi - \frac{1}{Re_1} \Delta v - \frac{1}{Re_2} \frac{\partial^2 v}{\partial \xi^2} = 0, \tag{2.1}$$

$$\operatorname{div} v + \frac{\partial \omega}{\partial \xi} = 0, \tag{2.2}$$

$$\frac{\partial \Phi}{\partial \xi} + \frac{bP}{p}(1+aq)T = 0, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + \nabla_v T + \omega \frac{\partial T}{\partial \xi} - \frac{bP}{p}(1+aq)\omega - \frac{1}{Rt_1} \Delta T - \frac{1}{Rt_2} \frac{\partial^2 T}{\partial \xi^2} = Q_1, \tag{2.4}$$

$$\frac{\partial q}{\partial t} + \nabla_v q + \omega \frac{\partial q}{\partial \xi} - \frac{1}{Rq_1} \Delta q - \frac{1}{Rq_2} \frac{\partial^2 q}{\partial \xi^2} = Q_2, \tag{2.5}$$

where the unknown functions v, ω, Φ, q, T are: $v = (v_\theta, v_\varphi)$ the horizontal velocity, ω vertical velocity in p-coordinate system, Φ the geopotential, q the mixing ratio of water vapor in the air, T temperature, $f = 2\cos\theta$ Coriolis parameter, R_0 the Rossby number, k vertical unit vector, Re_1, Re_2, Rt_1, Rt_2, Rq_1, Rq_2 Reynolds numbers, P an approximate value of pressure at the surface of the earth, p_0 pressure of the upper atmosphere and $p_0 > 0$, the variable ξ satisfying $p = (P - p_0)\xi + p_0$ ($0 < p_0 \leqslant p \leqslant P$), Q_1, Q_2 are given functions on $S^2 \times (0,1)$ (here we don't consider the condensation of water vapor), a a positive constant ($a \approx 0.618$), b a positive constant. The definitions of $\nabla_v v$, Δv, ΔT, Δq, $\nabla_v q$, $\nabla_v T$, $\operatorname{div} v$, $\operatorname{grad}\Phi$ will be given in Section 4.1. Eqs. (2.1)–(2.5) are called the 3-D viscous primitive equations of the large-scale moist atmosphere.

The space domain of Eqs. (2.1)–(2.5) is

$$\Omega = S^2 \times (0,1),$$

where S^2 is two-dimensional unit sphere. The boundary value conditions are given by

$$\xi = 1\ (p = P): \quad \frac{\partial v}{\partial \xi} = 0, \quad \omega = 0, \quad \frac{\partial T}{\partial \xi} = \alpha_s (T_s - T), \quad \frac{\partial q}{\partial \xi} = \beta_s (q_s - q), \tag{2.6}$$

$$\xi = 0\ (p = p_0):\quad \frac{\partial v}{\partial \xi} = 0,\quad \omega = 0,\quad \frac{\partial T}{\partial \xi} = 0,\quad \frac{\partial q}{\partial \xi} = 0, \tag{2.7}$$

where α_s, β_s are positive constants, T_s the given temperature on the surface of the earth, q_s the given mixing ratio of water vapor on the surface of the earth. For simplicity and without loss of generality we assume that $T_s = 0$ and $q_s = 0$. If $T_s \neq 0$ and $q_s \neq 0$, one can homogenize the boundary value conditions for T, q (cf., e.g., [16]).

Integrating (2.2) and using the boundary conditions (2.6), (2.7), we have

$$\omega(t;\theta,\varphi,\xi) = W(v)(t;\theta,\varphi,\xi) = \int_\xi^1 \operatorname{div} v\big(t;\theta,\varphi,\xi'\big)\,d\xi', \tag{2.8}$$

$$\int_0^1 \operatorname{div} v\,d\xi = 0. \tag{2.9}$$

Suppose that Φ_s is a certain unknown function at the isobaric surface $\xi = 1$. Integrating (2.3), we obtain

$$\Phi(t;\theta,\varphi,\xi) = \Phi_s(t;\theta,\varphi) + \int_\xi^1 \frac{bP}{p}(1+aq)T\,d\xi'. \tag{2.10}$$

In this article, we assume that the constants Re_1, Re_2, Rt_1, Rt_2, Rq_1, Rq_2 are all equal to 1, which cannot change our results. Then Eqs. (2.1)–(2.5) can be written as

$$\frac{\partial v}{\partial t} + \nabla_v v + W(v)\frac{\partial v}{\partial \xi} + \frac{f}{R_0}k\times v + \operatorname{grad}\Phi_s + \int_\xi^1 \frac{bP}{p}\operatorname{grad}\big[(1+aq)T\big]\,d\xi'$$

$$- \Delta v - \frac{\partial^2 v}{\partial \xi^2} = 0, \tag{2.11}$$

$$\frac{\partial T}{\partial t} + \nabla_v T + W(v)\frac{\partial T}{\partial \xi} - \frac{bP}{p}(1+aq)W(v) - \Delta T - \frac{\partial^2 T}{\partial \xi^2} = Q_1, \tag{2.12}$$

$$\frac{\partial q}{\partial t} + \nabla_v q + W(v)\frac{\partial q}{\partial \xi} - \Delta q - \frac{\partial^2 q}{\partial \xi^2} = Q_2, \tag{2.13}$$

$$\int_0^1 \operatorname{div} v\,d\xi = 0, \tag{2.14}$$

where the definitions of $\operatorname{grad}[(1+aq)T]$, $\operatorname{grad}\Phi_s$ will be given in Section 4.1. The boundary value conditions of Eqs. (2.11)–(2.14) are given by

$$\xi = 1:\quad \frac{\partial v}{\partial \xi} = 0,\quad \frac{\partial T}{\partial \xi} = -\alpha_s T,\quad \frac{\partial q}{\partial \xi} = -\beta_s q, \tag{2.15}$$

$$\xi = 0:\quad \frac{\partial v}{\partial \xi} = 0,\quad \frac{\partial T}{\partial \xi} = 0,\quad \frac{\partial q}{\partial \xi} = 0; \tag{2.16}$$

and the initial value conditions can be given as

$$U|_{t=0} = (v|_{t=0}, T|_{t=0}, q|_{t=0}) = U_0 = (v_0, T_0, q_0). \tag{2.17}$$

We call (2.11)–(2.17) the initial boundary value problem of the new formulation of the 3-D viscous primitive equations of large-scale moist atmosphere, which is denoted by (IBVP).

Now we define the fluctuation $\tilde{v}$ of horizontal velocity and find the equations satisfied by $\tilde{v}$ and $\bar{v}$. By integrating the momentum equation (2.11) with respect to ξ from 0 to 1 and using the boundary value conditions (2.15) and (2.16), we get

$$\begin{aligned}&\frac{\partial \bar{v}}{\partial t} + \int_0^1 \left(\nabla_v v + W(v)\frac{\partial v}{\partial \xi}\right) d\xi + \frac{f}{R_0} k \times \bar{v} + \operatorname{grad} \Phi_s + \int_0^1 \int_\xi^1 \frac{bP}{p} \operatorname{grad}\big[(1+aq)T\big]\, d\xi'\, d\xi \\ &\quad - \Delta \bar{v} = 0 \quad \text{in } S^2,\end{aligned} \tag{2.18}$$

where $\bar{v} = \int_0^1 v\, d\xi$.

Denote the fluctuation of the horizontal velocity by

$$\tilde{v} = v - \bar{v}.$$

We notice that

$$\bar{\tilde{v}} = \int_0^1 \tilde{v}\, d\xi = 0, \qquad \nabla \cdot \bar{v} = 0. \tag{2.19}$$

By integration by parts and (2.19), we have

$$\int_0^1 W(v)\frac{\partial v}{\partial \xi}\, d\xi = \int_0^1 v \operatorname{div} v\, d\xi = \int_0^1 \tilde{v} \operatorname{div} \tilde{v}\, d\xi, \tag{2.20}$$

$$\int_0^1 \nabla_v v\, d\xi = \int_0^1 \nabla_{\tilde{v}} \tilde{v}\, d\xi + \nabla_{\bar{v}} \bar{v}. \tag{2.21}$$

From (2.18), (2.20) and (2.21), we obtain

$$\begin{aligned}&\frac{\partial \bar{v}}{\partial t} + \nabla_{\bar{v}} \bar{v} + \overline{\tilde{v} \operatorname{div} \tilde{v} + \nabla_{\tilde{v}} \tilde{v}} + \frac{f}{R_0} k \times \bar{v} + \operatorname{grad} \Phi_s + \int_0^1 \int_\xi^1 \frac{bP}{p} \operatorname{grad}\big[(1+aq)T\big]\, d\xi'\, d\xi \\ &\quad - \Delta \bar{v} = 0 \quad \text{in } S^2.\end{aligned} \tag{2.22}$$

Subtracting (2.22) from (2.11), we know that the fluctuation $\tilde{v}$ satisfies the following equation and boundary value conditions

$$\frac{\partial \tilde{v}}{\partial t} + \nabla_{\tilde{v}} \tilde{v} + \left(\int_\xi^1 \operatorname{div} \tilde{v} \, d\xi' \right) \frac{\partial \tilde{v}}{\partial \xi} + \nabla_{\tilde{v}} \bar{v} + \nabla_{\bar{v}} \tilde{v} - \overline{(\tilde{v} \operatorname{div} \tilde{v} + \nabla_{\tilde{v}} \tilde{v})} + \frac{f}{R_0} k \times \tilde{v}$$
$$+ \int_\xi^1 \frac{bP}{p} \operatorname{grad}\big[(1+aq)T\big] \, d\xi' - \int_0^1 \int_\xi^1 \frac{bP}{p} \operatorname{grad}\big[(1+aq)T\big] \, d\xi' \, d\xi$$
$$- \Delta \tilde{v} - \frac{\partial^2 \tilde{v}}{\partial \xi^2} = 0 \quad \text{in } \Omega, \tag{2.23}$$

$$\xi = 1: \quad \frac{\partial \tilde{v}}{\partial \xi} = 0, \tag{2.24}$$

$$\xi = 0: \quad \frac{\partial \tilde{v}}{\partial \xi} = 0. \tag{2.25}$$

3. Statements of main results

Now we formulate our main results in the present paper.

Proposition 3.1 *(Existence of global strong solutions for (IBVP)). Let Q_1, $Q_2 \in H^1(\Omega)$, $U_0 = (v_0, T_0, q_0) \in V$. Then for any $\mathcal{T} > 0$ given, there exists a strong solution U of the system* (2.11)–(2.17) *on the interval* $[0, \mathcal{T}]$, *where the definition of the space V will be given in Section* 4.1, *and the definition of strong solutions to the system* (2.11)–(2.17) *will be given in Section* 5.1.

Proposition 3.2 *(Uniqueness of global strong solutions for (IBVP)). Let $Q_1, Q_2 \in H^1(\Omega)$, $U_0 = (v_0, T_0, q_0) \in V$. Then for any $\mathcal{T} > 0$ given, the strong solution U of the system* (2.11)–(2.17) *on the interval* $[0, \mathcal{T}]$ *is unique. Moreover, the strong solution U is dependent continuously on the initial data.*

Proposition 3.3 *(Existence of bounded absorbing sets for the dynamical system (2.11)–(2.16)). If $Q_1, Q_2 \in H^1(\Omega)$, $U_0 = (v_0, T_0, q_0) \in V$, then the global strong solution U of the system* (2.11)–(2.17) *satisfies $U \in L^\infty(0, \infty; V)$ and*

$$\big\|U(t)\big\| \leqslant C\big(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1\big),$$

where C is a positive constant dependent on $\|U_0\|, \|Q_1\|_1, \|Q_2\|_1$ and $0 \leqslant t \leqslant +\infty$. Moreover, the corresponding semigroup $\{S(t)\}_{t \geqslant 0}$ possesses a bounded absorbing set B_ρ in V, i.e., for every bounded set $B \subset V$, there exists $t_0(B) > 0$ big enough such that

$$S(t)B \subset B_\rho, \quad \textit{for any } t \geqslant t_0,$$

where $B_\rho = \{U;\ \|U\| \leqslant \rho\}$ and ρ is a positive constant dependent on $\|Q_1\|_1, \|Q_2\|_1$.

Theorem 3.4 *(Existence of the universal attractor for the system (2.11)–(2.16)). The system* (2.11)–(2.16) *possesses a* (*weak*) *universal attractor $\mathcal{A} = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} S(t) B_\rho}$ that captures all the trajectories, where the closures are taken with respect to V-weak topology. The* (*weak*) *universal attractor $\mathcal{A}$ has the following properties*:

(i) (*weak compact*) *$\mathcal{A}$ is bounded and weakly closed in V*;
(ii) (*invariant*) *for every $t \geqslant 0$, $S(t)\mathcal{A} = \mathcal{A}$*;

(ii) (*attracting*) *for every bounded set B in V, the sets $S(t)B$ converge to $\mathcal{A}$ with respect to V-weak topology as $t \to +\infty$, i.e.,*

$$\lim_{t\to+\infty} d_V^w\big(S(t)B, \mathcal{A}\big) = 0,$$

where the distance d_V^w is induced by the V-weak topology.

Remark 3.5. The (weak) universal attractor $\mathcal{A}$ has the following additional properties:

(i) By the Rellich–Kondrachov Compact Embedding Theorem (cf., e.g., [1]), we know that for any $1 \leqslant p < 6$ the sets $S(t)B$ converge to $\mathcal{A}$ with respect to the $L^p(\Omega) \times L^p(\Omega) \times L^p(\Omega) \times L^p(\Omega)$-norm.
(ii) The (weak) universal attractor $\mathcal{A}$ is unique and is connected with respect to V-weak topology.

Remark 3.6. In comparison to the 3-D incompressible Navier–Stokes equations, the 3-D viscous primitive equations of large-scale moist atmosphere have not the time derivative term of the vertical velocity $\omega = W(v)$. Therefore, we cannot prove that the bounded absorbing set B_ρ in V is bounded in $H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)$ as in the case of 3-D incompressible Navier–Stokes equations (for the Navier–Stokes equations, if there exists a bounded absorbing set B_ρ in $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, then one can prove that B_ρ is bounded in $H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$), i.e., we cannot prove that the universal attractor $\mathcal{A}$ is compact in V.

4. Preliminaries

4.1. Some function spaces

Let e_θ, e_φ, e_ξ be the unit vectors in θ, φ and ξ directions of the space domain Ω respectively,

$$e_\theta = \frac{\partial}{\partial\theta}, \qquad e_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}, \qquad e_\xi = \frac{\partial}{\partial\xi}.$$

The inner product and norm on $T_{(\theta,\varphi,\xi)}\Omega$ (the tangent space of Ω at the point (θ,φ,ξ)) are given by

$$(X,Y) = X\cdot Y = X_1Y_1 + X_2Y_2 + X_3Y_3, \qquad |X| = (X,X)^{\frac{1}{2}}$$

for

$$X = X_1e_\theta + X_2e_\varphi + X_3e_\xi, \qquad Y = Y_1e_\theta + Y_2e_\varphi + Y_3e_\xi \in T_{(\theta,\varphi,\xi)}\Omega.$$

$L^p(\Omega) := \{h;\ h:\Omega \to \mathbb{R},\ \int_\Omega |h|^p < +\infty\}$ with the norm $|h|_p = (\int_\Omega |h|^p)^{\frac{1}{p}}$, $1 \leqslant p < \infty$. $\int_\Omega \cdot\, d\Omega$ and $\int_{S^2} \cdot\, dS^2$ are denoted by $\int_\Omega \cdot$ and $\int_{S^2} \cdot$ respectively. $L^2(T\Omega|TS^2)$ is the first two components of L^2 vector fields on Ω with the norm $|v|_2 = (\int_\Omega (|v_\theta|^2 + |v_\varphi|^2))^{\frac{1}{2}}$, where $v = (v_\theta, v_\varphi): \Omega \to TS^2$. $C^\infty(S^2)$ is the function space for all smooth functions from S^2 to $\mathbb{R}$. $C^\infty(\Omega)$ is the function space for all smooth functions from Ω to $\mathbb{R}$. $C^\infty(T\Omega|TS^2)$ is the first two components of smooth vector fields on Ω. $H^m(\Omega)$ is the Sobolev space of functions which are in L^2, together with all their covariant derivatives with respect to e_θ, e_φ, e_ξ of order $\leqslant m$, with the norm

$$\|h\|_m = \left[\int_\Omega \left(\sum_{1\leqslant k\leqslant m}\ \sum_{i_j=1,2,3;\, j=1,\dots,k} |\nabla_{i_1}\cdots\nabla_{i_k} h|^2 + |h|^2\right)\right]^{\frac{1}{2}},$$

where $\nabla_1 = \nabla_{e_\theta}$, $\nabla_2 = \nabla_{e_\varphi}$, $\nabla_3 = \frac{\partial}{\partial \xi}$ (the definitions of $\nabla_{e_\theta}, \nabla_{e_\varphi}$ will be given later). $H^m(T\Omega|TS^2) = \{v;\ v = (v_\theta, v_\varphi) : \Omega \to TS^2,\ \|v\|_m^m < +\infty\}$ is the norm which is similar to that of $H^m(\Omega)$, that is, in the above formula of norm, we let $h = (v_\theta, v_\varphi) = v_\theta e_\theta + v_\varphi e_\varphi$.

The horizontal divergence div, the horizontal gradient $\nabla = \mathrm{grad}$, the horizontal covariant derivative ∇_v and the horizontal Laplace–Beltrami operator Δ for scalar and vector functions are defined by

$$\operatorname{div} v = \operatorname{div}(v_\theta e_\theta + v_\varphi e_\varphi) = \frac{1}{\sin\theta}\left(\frac{\partial v_\theta \sin\theta}{\partial\theta} + \frac{\partial v_\varphi}{\partial\varphi}\right), \tag{4.1}$$

$$\nabla T = \operatorname{grad} T = \frac{\partial T}{\partial\theta} e_\theta + \frac{1}{\sin\theta}\frac{\partial T}{\partial\varphi} e_\varphi, \tag{4.2}$$

$$\operatorname{grad} \Phi_s = \frac{\partial \Phi_s}{\partial\theta} e_\theta + \frac{1}{\sin\theta}\frac{\partial \Phi_s}{\partial\varphi} e_\varphi, \tag{4.3}$$

$$\nabla_v \tilde v = \left(v_\theta \frac{\partial \widetilde{v_\theta}}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial \widetilde{v_\theta}}{\partial\varphi} - v_\varphi \widetilde{v_\varphi}\cot\theta\right) e_\theta + \left(v_\theta \frac{\partial \widetilde{v_\varphi}}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial \widetilde{v_\varphi}}{\partial\varphi} + v_\varphi \widetilde{v_\theta}\cot\theta\right) e_\varphi, \tag{4.4}$$

$$\nabla_v T = v_\theta \frac{\partial T}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial T}{\partial\varphi}, \tag{4.5}$$

$$\nabla_v q = v_\theta \frac{\partial q}{\partial\theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial q}{\partial\varphi}, \tag{4.6}$$

$$\Delta T = \operatorname{div}(\operatorname{grad} T) = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial T}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\varphi^2}\right], \tag{4.7}$$

$$\Delta q = \operatorname{div}(\operatorname{grad} q) = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial q}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 q}{\partial\varphi^2}\right], \tag{4.8}$$

$$\Delta v = \left(\Delta v_\theta - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\theta}{\sin^2\theta}\right) e_\theta + \left(\Delta v_\varphi + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{\sin^2\theta}\right) e_\varphi, \tag{4.9}$$

where $v = v_\theta e_\theta + v_\varphi e_\varphi$, $\tilde v = \widetilde{v_\theta} e_\theta + \widetilde{v_\varphi} e_\varphi \in C^\infty(T\Omega|TS^2)$, $T, q \in C^\infty(\Omega)$, $\Phi_s \in C^\infty(S^2)$.

Now we can define our working spaces for the problem (IBVP). Let

$$\widetilde{\mathcal{V}_1} := \left\{v;\ v \in C^\infty\big(T\Omega|TS^2\big),\ \frac{\partial v}{\partial\xi}\bigg|_{\xi=0} = 0,\ \frac{\partial v}{\partial\xi}\bigg|_{\xi=1} = 0,\ \int_0^1 \operatorname{div} v \, d\xi = 0\right\},$$

$$\widetilde{\mathcal{V}_2} := \left\{T;\ T \in C^\infty(\Omega),\ \frac{\partial T}{\partial\xi}\bigg|_{\xi=0} = 0,\ \frac{\partial T}{\partial\xi}\bigg|_{\xi=1} = -\alpha_s T\right\},$$

$$\widetilde{\mathcal{V}_3} := \left\{q;\ q \in C^\infty(\Omega),\ \frac{\partial q}{\partial\xi}\bigg|_{\xi=0} = 0,\ \frac{\partial q}{\partial\xi}\bigg|_{\xi=1} = -\beta_s q\right\},$$

$$V_1 = \text{the closure of } \widetilde{\mathcal{V}_1} \text{ with respect to the norm } \|\cdot\|_1,$$

$$V_2 = \text{the closure of } \widetilde{\mathcal{V}_2} \text{ with respect to the norm } \|\cdot\|_1,$$

$$V_3 = \text{the closure of } \widetilde{\mathcal{V}_3} \text{ with respect to the norm } \|\cdot\|_1,$$

$$H_1 = \text{the closure of } \widetilde{\mathcal{V}_1} \text{ with respect to the norm } |\cdot|_2,$$

$$H_2 = L^2(\Omega),$$

$$V = V_1 \times V_2 \times V_3,$$
$$H = H_1 \times H_2 \times H_2.$$

The inner products and norms on V_1, V_2, V_3 are given by

$$(v, v_1)_{V_1} = \int_\Omega \left(\nabla_{e_\theta} v \cdot \nabla_{e_\theta} v_1 + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} v_1 + \frac{\partial v}{\partial \xi} \frac{\partial v_1}{\partial \xi} + v \cdot v_1 \right),$$
$$\|v\| = (v, v)_{V_1}^{\frac{1}{2}}, \quad \forall v, v_1 \in V_1,$$
$$(T, T_1)_{V_2} = \int_\Omega \left(\operatorname{grad} T \cdot \operatorname{grad} T_1 + \frac{\partial T}{\partial \xi} \frac{\partial T_1}{\partial \xi} + T T_1 \right),$$
$$\|T\| = (T, T)_{V_2}^{\frac{1}{2}}, \quad \forall T, T_1 \in V_2,$$
$$(q, q_1)_{V_3} = \int_\Omega \left(\operatorname{grad} q \cdot \operatorname{grad} q_1 + \frac{\partial q}{\partial \xi} \frac{\partial q_1}{\partial \xi} + q q_1 \right),$$
$$\|q\| = (q, q)_{V_3}^{\frac{1}{2}}, \quad \forall q, q_1 \in V_2,$$
$$(U, U_1)_H = (v, v_1) + (T, T_1) + (q, q_1),$$
$$(U, U_1)_V = (v, v_1)_{V_1} + (T, T_1)_{V_2} + (q, q_1)_{V_3},$$
$$\|U\| = (U, U)_V^{\frac{1}{2}}, \qquad |U|_2 = (U, U)_H^{\frac{1}{2}}, \quad \forall U = (v, T, q), U_1 = (v_1, T_1, q_1) \in V,$$

where $(\cdot,\cdot)$ denotes the L^2 inner products in H_1, H_2.

4.2. Some lemmas

Lemma 4.1. *Let $u = (u_\theta, u_\varphi)$, $u_1 = ((u_1)_\theta, (u_1)_\varphi) \in C^\infty(T\Omega|TS^2)$, and $p \in C^\infty(S^2)$. Then*

(1)

$$\int_{S^2} p \operatorname{div} u = -\int_{S^2} \nabla p \cdot u, \tag{4.10}$$

in particular,

$$\int_{S^2} \nabla p \cdot v = 0 \quad \textit{for any } v \in V, \tag{4.11}$$

(2)

$$\int_\Omega (-\Delta u) \cdot u_1 = \int_\Omega (\nabla_{e_\theta} u \cdot \nabla_{e_\theta} u_1 + \nabla_{e_\varphi} u \cdot \nabla_{e_\varphi} u_1 + u \cdot u_1). \tag{4.12}$$

Proof. We can prove (4.10) by using (4.1), (4.2) and the Stokes Theorem (cf., e.g., [32,36]). (4.11) is the direct result of (4.10) and (2.14). From (4.4) and (4.9), by direct computation, we can obtain the second part. □

Lemma 4.2. *For any $h \in C^\infty(S^2)$, $v \in C^\infty(T\Omega|TS^2)$, we have*

$$\int_{S^2} \nabla_v h + \int_{S^2} h \operatorname{div} v = \int_{S^2} \operatorname{div}(hv) = 0.$$

Proof. From (4.1), (4.5), (4.11), by direct computation, we can prove Lemma 4.2. □

Lemma 4.3. *Let v, $v_1 \in V_1$, $T \in V_2$, $q \in V_3$. Then we have*

1) $\int_\Omega [\nabla_v v_1 + (\int_\xi^1 \operatorname{div} v \, d\xi') \frac{\partial v_1}{\partial \xi}] v_1 = 0$,
2) $\int_\Omega [\nabla_v g + (\int_\xi^1 \operatorname{div} v \, d\xi') \frac{\partial g}{\partial \xi}] g = 0$ *for* $g = T$ *or* $g = q$,
3) $\int_\Omega \{\int_\xi^1 \frac{bP}{p} \operatorname{grad}[(1+aq)T] \, d\xi' \cdot v - \frac{bP}{p}(1+aq) W(v) \cdot T\} = 0$.

For the details of the proof for Lemma 4.3, we refer the reader to [16, Lemma 3.2].

For convenience, we recall some interpolation inequalities (for the details of the proof, we refer the reader to see, e.g., [1,15,31]).

i) For $u \in H^1(S^2)$ (for the definitions of $H^1(S^2)$, $L^p(S^2)$, cf. [24]),

$$\|u\|_{L^4} \leqslant c\|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}}, \tag{4.13}$$

$$\|u\|_{L^6} \leqslant c\|u\|_{L^4}^{\frac{2}{3}} \|u\|_{H^1}^{\frac{1}{3}}, \tag{4.14}$$

$$\|u\|_{L^8} \leqslant c\|u\|_{L^4}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}}, \tag{4.15}$$

where $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(S^2)}$ for $1 \leqslant p < +\infty$, $\|\cdot\|_{H^1} = \|u\|_{H^1(S^2)}$.

ii) For $u \in H^1(\Omega)$,

$$|u|_4 \leqslant c|u|_2^{\frac{1}{4}} \|u\|_1^{\frac{3}{4}}. \tag{4.16}$$

5. A priori estimates about local strong solutions

At first, we recall the local, in time, existence of strong solutions of the 3-D viscous primitive equations of the large-scale moist atmosphere.

Definition 5.1. Let $U_0 = (v_0, T_0, q_0) \in V$, and let $\mathcal{T}$ be a fixed positive time. $U = (v, T, q)$ is called a strong solution of the system (2.11)–(2.17) on the time interval $[0, \mathcal{T}]$ if it satisfies (2.11)–(2.13) in weak sense such that

$$v \in C([0,\mathcal{T}]; V_1) \cap L^2(0,\mathcal{T}; (H^2(\Omega))^2),$$
$$T \in C([0,\mathcal{T}]; V_2) \cap L^2(0,\mathcal{T}; H^2(\Omega)),$$
$$q \in C([0,\mathcal{T}]; V_3) \cap L^2(0,\mathcal{T}; H^2(\Omega)),$$
$$\frac{\partial v}{\partial t} \in L^1(0,\mathcal{T}; (L^2(\Omega))^2),$$
$$\frac{\partial T}{\partial t}, \frac{\partial q}{\partial t} \in L^1(0,\mathcal{T}; L^2(\Omega)).$$

Remark 5.2. Since the 3-D viscous primitive equations of large-scale moist atmosphere have not the time derivative term of the vertical velocity $\omega = W(v)$, we cannot prove $\frac{\partial v}{\partial t} \in L^2(0, \mathcal{T}; (L^2(\Omega))^2)$, $\frac{\partial T}{\partial t}, \frac{\partial q}{\partial t} \in L^2(0, \mathcal{T}; L^2(\Omega))$.

Proposition 5.3. *Let $Q_1, Q_2 \in H^1(\Omega)$, $U_0 = (v_0, T_0, q_0) \in V$. Then there exists $\mathcal{T}_* > 0$, $\mathcal{T}_* = \mathcal{T}_*(\|U_0\|)$, and there exists a strong solution U of the system* (2.11)–(2.17) *on the interval* $[0, \mathcal{T}_*]$.

Proof. The proof of Proposition 5.3 is similar to that for the primitive equations of large-scale ocean given in the papers [18,34]. So we omit the details of the proof here. $\square$

Remark 5.4. In order to prove the global, in time, existence of strong solutions to the system (2.11)–(2.17) and study the long-time behavior of strong solutions, we should make a priori estimates about H^1-norm of the local solution $U(t)$ obtained in Proposition 5.3. In this section, we should show that **H^1-norm of the strong solution $U(t)$ is bounded uniformly in t**.

Remark 5.5. In order to study the long-time behavior of strong solutions, we must make three key estimates. First, we must make estimates about L^3-norm of the temperature T and the fluctuation $\tilde{v}$ of horizontal velocity v before we study the long-time behavior of strong solutions by the Uniform Gronwall Lemma, without which we only obtain the global well-posedness of (IBVP). Second, on the basis of L^3 estimates of $\tilde{v}, T$, we ought to make estimates about L^4-norm of $\tilde{v}$, T and the mixing ratio of water vapor in the air q. If we only made estimates about L^6-norm of $\tilde{v}$, q, T as that in [11], we could only prove the global well-posedness for the problem (IBVP), but we could not study long-time behavior of strong solutions and could not obtain a stronger result than the uniqueness of strong solutions to (IBVP). Third, since the moist atmospheric equations are more complicated than the oceanic primitive equations, we have to make estimates about $\partial_\xi T$, $\partial_\xi q$ before we prove that H^1-norm of v, T, q is bounded.

5.1. L^2 estimates of v, T, q

Choosing v as a test function in Eq. (2.11), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d|v|_2^2}{dt} + \int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2\right) + \int_\Omega \left|\frac{\partial v}{\partial \xi}\right|^2 \\
&\quad = -\int_\Omega \left(\nabla_v v + W(v)\frac{\partial v}{\partial \xi} + \frac{f}{R_0} k \times v + \operatorname{grad} \Phi_s\right) \cdot v \\
&\qquad - \int_\Omega \left\{\int_\xi^1 \frac{bP}{p} \operatorname{grad}\left[(1+aq)T\right] d\xi'\right\} \cdot v.
\end{aligned} \tag{5.1}$$

By Lemma 4.1, Lemma 4.3 and $(\frac{f}{R_0} k \times v) \cdot v = 0$, we obtain from (5.1)

$$\begin{aligned}
&\frac{1}{2}\frac{d|v|_2^2}{dt} + \int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2\right) + \int_\Omega \left|\frac{\partial v}{\partial \xi}\right|^2 \\
&\quad = -\int_\Omega \left\{\int_\xi^1 \frac{bP}{p} \operatorname{grad}\left[(1+aq)T\right] d\xi'\right\} \cdot v.
\end{aligned} \tag{5.2}$$

Taking the inner product of Eq. (2.12) with T in $L^2(\Omega)$, by Lemma 4.3, we obtain

$$\frac{1}{2}\frac{d|T|_2^2}{dt}+\int_\Omega |\nabla T|^2+\int_\Omega \left|\frac{\partial T}{\partial \xi}\right|^2+\alpha_s|T|_{\xi=1}|_2^2 = \int_\Omega \frac{bP}{p}(1+aq)TW(v)+\int_\Omega Q_1 T. \tag{5.3}$$

Similarly to (5.3), we have

$$\frac{1}{2}\frac{d|q|_2^2}{dt}+\int_\Omega |\nabla q|^2+\int_\Omega \left|\frac{\partial q}{\partial \xi}\right|^2+\beta_s|q|_{\xi=1}|_2^2=\int_\Omega qQ_2. \tag{5.4}$$

From (5.1)–(5.4), by Lemma 4.3, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d(|v|_2^2+|T|_2^2+|q|_2^2)}{dt}+\int_\Omega \big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2+|v|^2\big)+\int_\Omega \left|\frac{\partial v}{\partial \xi}\right|^2\\ &\quad+\int_\Omega |\nabla T|^2+\int_\Omega \left|\frac{\partial T}{\partial \xi}\right|^2+\alpha_s|T|_{\xi=1}|_2^2+\int_\Omega |\nabla q|^2+\int_\Omega \left|\frac{\partial q}{\partial \xi}\right|^2+\beta_s|q|_{\xi=1}|_2^2\\ &=\int_\Omega Q_1 T+\int_\Omega qQ_2.\end{aligned} \tag{5.5}$$

By $T(\theta,\varphi,\xi)=-\int_\xi^1 \frac{\partial T}{\partial \xi'}\,d\xi'+T|_{\xi=1}$, $q(\theta,\varphi,\xi)=-\int_\xi^1 \frac{\partial q}{\partial \xi'}\,d\xi'+q|_{\xi=1}$, using the Hölder inequality and the Cauchy–Schwarz inequality, we have

$$|T|_2^2\leqslant 2\left|\frac{\partial T}{\partial \xi}\right|_2^2+2|T|_{\xi=1}|_2^2,\qquad |q|_2^2\leqslant 2\left|\frac{\partial q}{\partial \xi}\right|_2^2+2|q|_{\xi=1}|_2^2. \tag{5.6}$$

From (5.5)–(5.6), by the Young inequality with ε, we obtain

$$\begin{aligned}&\frac{d(|v|_2^2+|T|_2^2+|q|_2^2)}{dt}+\int_\Omega \big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2+|v|^2\big)+\int_\Omega \left|\frac{\partial v}{\partial \xi}\right|^2\\ &\quad+\int_\Omega |\nabla T|^2+\int_\Omega \left|\frac{\partial T}{\partial \xi}\right|^2+\alpha_s|T|_{\xi=1}|_2^2+\int_\Omega |\nabla q|^2+\int_\Omega \left|\frac{\partial q}{\partial \xi}\right|^2+\beta_s|q|_{\xi=1}|_2^2\\ &\leqslant c\int_\Omega \big(Q_1^2+Q_2^2\big).\end{aligned} \tag{5.7}$$

In this article, c will denote positive constant and can be determined in concrete conditions. ε is a small enough positive constant. By (5.6), (5.7) and thanks to the Gronwall inequality, we have

$$|v|_2^2+|T|_2^2+|q|_2^2\leqslant e^{-c_0 t}\big(|v_0|_2^2+|T_0|_2^2+|q_0|_2^2\big)+c\big(|Q_1|_2^2+|Q_2|_2^2\big)\leqslant E_0, \tag{5.8}$$

where c_0, E_0 are positive constants and $t \geqslant 0$. By the Minkowski inequality and the Hölder inequality, for any $t \geqslant 0$, we have

$$\left\|\bar{v}(t)\right\|_{L^2}^2 \leqslant \left|v(t)\right|_2^2 \leqslant e^{-c_0 t}\left(|v_0|_2^2+|T_0|_2^2+|q_0|_2^2\right)+c\left(|Q_1|_2^2+|Q_2|_2^2\right) \leqslant E_0. \tag{5.9}$$

From (5.6)–(5.8), we get

$$\begin{aligned}
&c_1 \int_t^{t+r}\left[\int_\Omega \left(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2+\left|\frac{\partial v}{\partial \xi}\right|^2\right)+\int_\Omega\left(|\nabla T|^2+\left|\frac{\partial T}{\partial \xi}\right|^2+|T|^2\right)\right.\\
&\qquad\left.+\int_\Omega\left(|\nabla q|^2+\left|\frac{\partial q}{\partial \xi}\right|^2+|q|^2\right)+|T|_{\xi=1}|_2^2+|q|_{\xi=1}|_2^2\right]+\left|U(t)\right|_2^2\\
&\quad\leqslant 2e^{-c_0 t}\left(|v_0|_2^2+|T_0|_2^2+|q_0|_2^2\right)+c\left(|Q_1|_2^2+|Q_2|_2^2\right)(2+r) \leqslant E_1,
\end{aligned} \tag{5.10}$$

where c_1, E_1 are positive constants, $t \geqslant 0$, $1 \geqslant r > 0$ are given, and $\int_t^{t+r} \cdot\, ds$ is denoted by $\int_t^{t+r} \cdot$. Since

$$\int_{S^2}\left(|\nabla_{e_\theta}\bar{v}|^2+|\nabla_{e_\varphi}\bar{v}|^2\right) \leqslant \int_\Omega\left(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\right),$$

from (5.10), we have

$$c_1 \int_t^{t+r}\int_{S^2}\left(|\nabla_{e_\theta}\bar{v}|^2+|\nabla_{e_\varphi}\bar{v}|^2\right)+\|\bar{v}\|_{L^2}^2 \leqslant E_1, \quad \forall t \geqslant 0. \tag{5.11}$$

5.2. L^4 estimates of q

By taking the inner product of Eq. (2.13) with $|q|^2 q$ in $L^2(\Omega)$, we get

$$\begin{aligned}
&\frac{1}{4}\frac{d|q|_4^4}{dt}+3\int_\Omega |\nabla q|^2 q^2+3\int_\Omega\left|\frac{\partial q}{\partial \xi}\right|^2 q^2+\beta_s\int_{S^2}|q|_{\xi=1}|^4\\
&\quad=\int_\Omega Q_2|q|^2 q-\int_\Omega\left[\nabla_v q+\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)\frac{\partial q}{\partial \xi}\right]|q|^2 q.
\end{aligned} \tag{5.12}$$

By Lemma 4.2, we have

$$\begin{aligned}
&\int_\Omega\left[\nabla_v q+\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)\frac{\partial q}{\partial \xi}\right]|q|^2 q\\
&\quad=\frac{1}{4}\int_\Omega \nabla_v q^4+\int_{S^2}\left[\int_0^1\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right) d\left(\frac{1}{4}q^4\right)\right]\\
&\quad=\frac{1}{4}\int_\Omega\left(\nabla_v q^4+q^4 \operatorname{div} v\right)=0.
\end{aligned} \tag{5.13}$$

Combining (5.12) with (5.13), we obtain

$$\frac{1}{4}\frac{d|q|_4^4}{dt}+3\int_\Omega |\nabla q|^2q^2+3\int_\Omega \left|\frac{\partial q}{\partial \xi}\right|^2 q^2+\beta_s\int_{S^2}|q|_{\xi=1}|^4=\int_\Omega Q_2|q|^2q. \tag{5.14}$$

Since $q^4(\theta,\varphi,\xi)=-\int_\xi^1 \frac{\partial q^4}{\partial \xi'}\,d\xi'+q^4|_{\xi=1}$, by using the Hölder inequality and the Cauchy–Schwarz inequality, we get

$$|q|_4^4\leqslant c\left(\int_\Omega |q|^2\left|\frac{\partial q}{\partial \xi}\right|^2\right)+\frac{1}{2}\int_\Omega q^4+|q|_{\xi=1}|_4^4. \tag{5.15}$$

From (5.14) and (5.15), by the Young inequality, we obtain

$$\frac{d|q|_4^4}{dt}+c_2|q|_4^4\leqslant c|Q_2|_4^4, \tag{5.16}$$

where c_2 is a positive constant. By the Gronwall inequality, we have

$$\left|q(t)\right|_4^4\leqslant e^{-c_2t}|q_0|_4^4+c|Q_2|_4^4\leqslant E_2, \tag{5.17}$$

where $t\geqslant 0$, E_2 is a positive constant. From (5.14) and (5.17), we get

$$c_1\int_t^{t+r}|q|_{\xi=1}|_4^4\leqslant 2E_2, \quad \text{for any } t\geqslant 0. \tag{5.18}$$

Before making L^3 and L^4 estimates of T by anisotropic estimates, we need the following lemma.

Lemma 5.6. *Let $v\in V_1$, $T\in V_2$. Then we have*

1) $\|\frac{bP}{p}\int_\xi^1 \operatorname{div} v\,d\xi'\|_{L^2_{(\theta,\varphi)}L^\infty_\xi}\leqslant|\operatorname{div} v|_2$,
2)

$$\left\|T^n\right\|_{L^2_{(\theta,\varphi)}L^1_\xi}\leqslant\begin{cases} c|T|_2\|T\|, & \text{if } n=2,\\ c|T|_4^2\|T\|, & \text{if } n=3,\end{cases}$$

where $\|\frac{bP}{p}\int_\xi^1 \operatorname{div} v\,d\xi'\|_{L^2_{(\theta,\varphi)}L^\infty_\xi}=\|[\int_{S^2}(\frac{bP}{p}\int_\xi^1 \operatorname{div} v\,d\xi')^2]^{\frac{1}{2}}\|_{L^\infty_\xi}$, $\|T^n\|_{L^2_{(\theta,\varphi)}L^1_\xi}=\|(\int_{S^2}|T^n|^2)^{\frac{1}{2}}\|_{L^1_\xi}$.

Proof. By the Hölder inequality, we can get 1). By (4.13), (4.14) and the Hölder inequality, we can obtain 2). □

5.3. L^3 estimates of T

We take the inner product of Eq. (2.12) with $|T|T$ in $L^2(\Omega)$ and obtain

$$\frac{1}{3}\frac{d|T|_3^3}{dt}+2\int_\Omega |\nabla T|^2|T|+2\int_\Omega \left|\frac{\partial T}{\partial \xi}\right|^2|T|+\alpha_s\int_{S^2}|T|_{\xi=1}|^3$$
$$=\int_\Omega Q_1|T|T-\int_\Omega\left[\nabla_v T+\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)\frac{\partial T}{\partial \xi}\right]|T|T$$
$$+\int_\Omega \frac{bP}{p}\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)|T|T+\int_\Omega \frac{abP}{p}\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)q|T|T. \tag{5.19}$$

By the Hölder inequality and Lemma 5.6, we get

$$\left|\int_\Omega \frac{bP}{p}\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)|T|T\right|$$
$$\leqslant \left\|\frac{bP}{p}\int_\xi^1 \operatorname{div} v\, d\xi'\right\|_{L^2_{(\theta,\varphi)}L^\infty_\xi}\left\|T^2\right\|_{L^2_{(\theta,\varphi)}L^1_\xi}$$
$$\leqslant c\int_\Omega \left(|\nabla_{e_\varphi} v|^2+|\nabla_{e_\varphi} v|^2\right)+c|T|_2^2\|T\|^2. \tag{5.20}$$

By the Hölder inequality, the Young inequality and $\|u\|_{L^{\frac{16}{3}}}\leqslant c\|u\|_{L^2}^{\frac{3}{8}}\|u\|_{H^1}^{\frac{5}{8}}$, for any $u\in H^1(S^2)$, we have

$$\left|\int_\Omega \frac{abP}{p}q\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)|T|T\right|$$
$$\leqslant c\int_0^1\left\{\left(\int_{S^2}q^4\right)^{\frac{1}{4}}\left[\int_{S^2}\left(|T|^{\frac{3}{2}}\right)^{\frac{16}{3}}\right]^{\frac{1}{4}}\right\}d\xi\left\|\frac{bP}{p}\int_\xi^1 \operatorname{div} v\, d\xi'\right\|_{L^2_{(\theta,\varphi)}L^\infty_\xi}$$
$$\leqslant c\int_0^1\left[\left(\int_{S^2}q^4\right)^{\frac{1}{4}}\|T\|_{L^3}^{\frac{3}{4}}\left\||T|^{\frac{3}{2}}\right\|_{H^1}^{\frac{5}{6}}\right]d\xi|\operatorname{div} v|_2$$
$$\leqslant c|q|_4|T|_3^{\frac{3}{4}}\left(\int_\Omega |T||\nabla T|^2+|T|_3^3\right)^{\frac{5}{12}}|\operatorname{div} v|_2$$
$$\leqslant c|q|_4^{\frac{12}{7}}\left(1+|T|_3^3\right)\left[1+\int_\Omega\left(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\right)\right]+\varepsilon\left(\int_\Omega |T||\nabla T|^2+|T|_3^3\right). \tag{5.21}$$

By the Young inequality, choosing ε small enough and using an inequality similar to (5.15), we derive from (5.19)–(5.21)

$$\frac{d|T|_3^3}{dt}+2\int_\Omega |\nabla T|^2|T|+2\int_\Omega\left|\frac{\partial T}{\partial\xi}\right|^2|T|+\alpha_s\int_{S^2}|T|_{\xi=1}|^3$$
$$\leqslant c|q|_4^{\frac{12}{7}}\left[1+\int_\Omega\big(|\nabla_{e_\theta}v|^2+|\nabla_{e_\varphi}v|^2\big)\right]|T|_3^3+c|Q_1|_3^3$$
$$+c\big(1+|q|_4^{\frac{12}{7}}\big)\left[1+\int_\Omega\big(|\nabla_{e_\theta}v|^2+|\nabla_{e_\varphi}v|^2\big)\right]+c|T|_2^2\|T\|^2. \tag{5.22}$$

By the Uniform Gronwall Lemma [32, p. 91], (5.10), (5.17) and $|T|_3^3\leqslant c|T|_2^{\frac{3}{2}}\|T\|^{\frac{3}{2}}$, we obtain

$$\big|T(t+r)\big|_3^3\leqslant E_3, \tag{5.23}$$

where $E_3=E_3(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$ and $t\geqslant 0$. By the Gronwall inequality, we prove

$$\big|T(t)\big|_3^3\leqslant C_0,\quad \text{for any } 0\leqslant t<r, \tag{5.24}$$

where $C_0=C_0(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$.

5.4. L^4 estimates of T

We take the inner product of Eq. (2.12) with $|T|^2T$ in $L^2(\Omega)$ and obtain

$$\frac{1}{4}\frac{d|T|_4^4}{dt}+3\int_\Omega|\nabla T|^2T^2+3\int_\Omega\left|\frac{\partial T}{\partial\xi}\right|^2T^2+\alpha_s\int_{S^2}|T|_{\xi=1}|^4$$
$$=\int_\Omega Q_1|T|^2T-\int_\Omega\left[\nabla_vT+\left(\int_\xi^1\operatorname{div}v\,d\xi'\right)\frac{\partial T}{\partial\xi}\right]|T|^2T$$
$$+\int_\Omega\frac{bP}{p}\left(\int_\xi^1\operatorname{div}v\,d\xi'\right)|T|^2T+\int_\Omega\frac{abP}{p}\left(\int_\xi^1\operatorname{div}v\,d\xi'\right)q|T|^2T. \tag{5.25}$$

Similarly to (5.20), by the Hölder inequality and Lemma 5.6, we get

$$\left|\int_\Omega\frac{bP}{p}\left(\int_\xi^1\operatorname{div}v\,d\xi'\right)|T|^2T\right|\leqslant c\int_\Omega\big(|\nabla_{e_\varphi}v|^2+|\nabla_{e_\varphi}v|^2\big)+c|T|_4^4\|T\|^2. \tag{5.26}$$

Similarly to (5.21), by the Hölder inequality, the Young inequality and $\|u\|_{L^6(S^2)}\leqslant c\|u\|_{L^2(S^2)}^{\frac{1}{3}}\|u\|_{H^1(S^2)}^{\frac{2}{3}}$, for any $u\in H^1(S^2)$, we have

$$\left|\int_\Omega \frac{abP}{p} q\left(\int_\xi^1 \operatorname{div} v\, d\xi'\right)|T|^2 T\right|$$

$$\leqslant c\big(|q|_4^4 + |T|_4^4\big)\int_\Omega \big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\big) + \varepsilon\left(\int_\Omega |T|^2|\nabla T|^2 + |T|_4^4\right). \tag{5.27}$$

By the Young inequality, (5.17), choosing ε small enough and using an inequality similar to (5.15), we derive from (5.25)–(5.27),

$$\frac{d|T|_4^4}{dt} + 3\int_\Omega |\nabla T|^2 T^2 + 3\int_\Omega \left|\frac{\partial T}{\partial \xi}\right|^2 T^2 + \alpha_s \int_{S^2} |T|_{\xi=1}|^4$$

$$\leqslant c\left[\|T\|^2 + \int_\Omega \big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\big)\right]|T|_4^4 + c|Q_1|_4^4 + c\|v\|^2. \tag{5.28}$$

By the Uniform Gronwall Lemma, (5.10), (5.23) and $|T|_4^4 \leqslant c|T|_3^2\|T\|^2$, we obtain

$$\big|T(t+2r)\big|_4^4 \leqslant E_4, \tag{5.29}$$

where $E_4 = E_4(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1) > 0$ and $t \geqslant 0$. By the Gronwall inequality, from (5.28) we prove

$$\big|T(t)\big|_4^4 \leqslant C_1, \tag{5.30}$$

where $C_1 = C_1(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1) > 0$ and $0 \leqslant t < 2r$. From (5.28) and (5.29), we get

$$c_1 \int_{t+2r}^{t+3r} |T|_{\xi=1}|_4^4 \leqslant E_4^2 + E_4, \quad \text{for any } t \geqslant 0. \tag{5.31}$$

5.5. L^3 estimates of $\tilde v$

We take the inner product of Eq. (2.23) with $|\tilde v|\tilde v$ in L^2 and obtain

$$\frac{1}{3}\frac{d|\tilde v|_3^3}{dt} + \int_\Omega \left[\big(|\nabla_{e_\theta}\tilde v|^2 + |\nabla_{e_\varphi}\tilde v|^2\big)|\tilde v| + \frac{4}{9}\big|\nabla_{e_\theta}|\tilde v|^{\frac{3}{2}}\big|^2 + \frac{4}{9}\big|\nabla_{e_\varphi}|\tilde v|^{\frac{3}{2}}\big|^2 + |\tilde v|^3\right]$$

$$+ \int_\Omega \left(|\tilde v_\xi|^2|\tilde v| + \frac{4}{9}\big|\partial_\xi|\tilde v|^{\frac{3}{2}}\big|^2\right)$$

$$= -\int_\Omega \left[\nabla_{\tilde v}\tilde v + \left(\int_\xi^1 \operatorname{div}\tilde v\, d\xi'\right)\frac{\partial \tilde v}{\partial \xi}\right]\cdot |\tilde v|\tilde v - \int_\Omega (\nabla_{\bar v}\tilde v)\cdot|\tilde v|\tilde v - \int_\Omega |\tilde v|\tilde v\cdot\nabla_{\tilde v}\bar v$$

$$-\int_\Omega \left\{\int_\xi^1 \frac{bP}{p}\operatorname{grad}\big[(1+aq)T\big]\,d\xi' - \int_0^1\int_\xi^1 \frac{bP}{p}\operatorname{grad}\big[(1+aq)T\big]d\xi'\,d\xi\right\}\cdot|\tilde v|\tilde v$$

$$+\int_\Omega \overline{(\tilde v \operatorname{div} \tilde v + \nabla_{\tilde v} \tilde v)} \cdot |\tilde v|\tilde v - \int_\Omega \left(\frac{f}{R_0} k \times \tilde v\right) \cdot |\tilde v|\tilde v, \tag{5.32}$$

where $\tilde v_\xi = \partial_\xi \tilde v$. Similarly to (5.13), by Lemma 4.2 and integration by parts, we have

$$\int_\Omega \left[\nabla_{\tilde v} \tilde v + \left(\int_\xi^1 \operatorname{div} \tilde v \, d\xi'\right) \frac{\partial \tilde v}{\partial \xi}\right] \cdot |\tilde v|\tilde v = 0. \tag{5.33}$$

By Lemma 4.2 and (2.19), we get

$$\int_\Omega (\nabla_{\bar v} \tilde v) \cdot |\tilde v|\tilde v = \frac{1}{3} \int_\Omega \nabla_{\bar v} |\tilde v|^3 = -\frac{1}{3} \int_\Omega |\tilde v|^3 \operatorname{div} \bar v = 0. \tag{5.34}$$

Using Lemma 4.2, we have

$$\begin{aligned} 0 = \int_\Omega \operatorname{div}\big[\big(|\tilde v|\tilde v \cdot \bar v\big)\tilde v\big] &= \int_\Omega \nabla_{\tilde v}\big(|\tilde v|\tilde v \cdot \bar v\big) + \int_\Omega |\tilde v|\tilde v \cdot \bar v \operatorname{div} \tilde v \\ &= \int_\Omega \big[|\tilde v|\tilde v \cdot \nabla_{\tilde v} \bar v + \bar v \cdot \nabla_{\tilde v}\big(|\tilde v|\tilde v\big)\big] + \int_\Omega |\tilde v|\tilde v \cdot \bar v \operatorname{div} \tilde v. \end{aligned}$$

So

$$-\int_\Omega |\tilde v|\tilde v \cdot \nabla_{\tilde v} \bar v = \int_\Omega \bar v \cdot \nabla_{\tilde v}\big(|\tilde v|\tilde v\big) + \int_\Omega |\tilde v|\tilde v \cdot \bar v \operatorname{div} \tilde v. \tag{5.35}$$

Using integration by parts, we obtain

$$\begin{aligned} \int_\Omega \left[\int_0^1 (\tilde v \operatorname{div} \tilde v + \nabla_{\tilde v} \tilde v) d\xi\right] \cdot |\tilde v|\tilde v &= \int_\Omega \left(\int_0^1 \tilde v_{e_\theta} \tilde v d\xi\right) \cdot \nabla_{e_\theta}\big(|\tilde v|\tilde v\big) \\ &\quad + \int_\Omega \left(\int_0^1 \tilde v_{e_\varphi} \tilde v d\xi\right) \cdot \nabla_{e_\varphi}\big(|\tilde v|\tilde v\big). \end{aligned} \tag{5.36}$$

From (5.32) to (5.36), by $(\frac{f}{R_0} k \times \tilde v) \cdot |\tilde v|\tilde v = 0$ and Lemma 4.1, we get

$$\begin{aligned} &\frac{1}{3} \frac{d|\tilde v|_3^3}{dt} + \int_\Omega \left[\big(|\nabla_{e_\theta} \tilde v|^2 + |\nabla_{e_\varphi} \tilde v|^2\big)|\tilde v| + \frac{4}{9}\big|\nabla_{e_\theta} |\tilde v|^{\frac{3}{2}}\big|^2 + \frac{4}{9}\big|\nabla_{e_\varphi} |\tilde v|^{\frac{3}{2}}\big|^2 + |\tilde v|^3\right] \\ &\quad + \int_\Omega \left(|\tilde v_\xi|^2 |\tilde v| + \frac{4}{9}\big|\partial_\xi |\tilde v|^{\frac{3}{2}}\big|^2\right) \\ &= \int_\Omega \big[\bar v \cdot \nabla_{\tilde v}\big(|\tilde v|\tilde v\big) + |\tilde v|\tilde v \cdot \bar v \operatorname{div} \tilde v\big] \end{aligned}$$

$$+\int_{\Omega}\left[\left(\int_0^1 \tilde{v}_\theta \tilde{v}\, d\xi\right)\cdot \nabla_{e_\theta}\big(|\tilde{v}|\tilde{v}\big)+\left(\int_0^1 \tilde{v}_\varphi \tilde{v}\, d\xi\right)\cdot \nabla_{e_\varphi}\big(|\tilde{v}|\tilde{v}\big)\right]$$

$$+\int_{\Omega}\left[\int_{\xi}^{1}\frac{bP}{p}(1+aq)T\, d\xi'-\int_0^1\int_{\xi}^{1}\frac{bP}{p}(1+aq)T\, d\xi'\, d\xi\right]\operatorname{div}\big(|\tilde{v}|\tilde{v}\big). \tag{5.37}$$

By the Hölder inequality, we derive from (5.37)

$$\frac{1}{3}\frac{d|\tilde{v}|_3^3}{dt}+\int_{\Omega}\left[\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)|\tilde{v}|+\frac{4}{9}\big|\nabla_{e_\theta}|\tilde{v}|^{\frac{3}{2}}\big|^2+\frac{4}{9}\big|\nabla_{e_\varphi}|\tilde{v}|^{\frac{3}{2}}\big|^2+|\tilde{v}|^3\right]$$

$$+\int_{\Omega}\left(|\tilde{v}_\xi|^2|\tilde{v}|+\frac{4}{9}\big|\partial_\xi|\tilde{v}|^{\frac{3}{2}}\big|^2\right)$$

$$\leqslant c\int_{S^2}|\bar{v}|\int_0^1|\tilde{v}|^2\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)^{\frac{1}{2}}\, d\xi$$

$$+c\int_{S^2}\left[\left(\int_0^1|\tilde{v}|^2\, d\xi\right)\int_0^1|\tilde{v}|\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)^{\frac{1}{2}}\, d\xi\right]$$

$$+c\int_{S^2}\left[\overline{|T|}\int_0^1|\tilde{v}|\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)^{\frac{1}{2}}\, d\xi\right]$$

$$+c\int_{S^2}\left[\overline{|qT|}\int_0^1|\tilde{v}|\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)^{\frac{1}{2}}\, d\xi\right]$$

$$\leqslant c\|\bar{v}\|_{L^4}\left[\int_{\Omega}|\tilde{v}|\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}\left[\int_{S^2}\left(\int_0^1|\tilde{v}|^3\, d\xi\right)^2\right]^{\frac{1}{4}}$$

$$+c\left[\int_{\Omega}|\tilde{v}|\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}\cdot\left(\int_{S^2}\left(\int_0^1|\tilde{v}|^2\, d\xi\right)^{\frac{5}{2}}\right)^{\frac{1}{2}}$$

$$+c\big\|\overline{|T|}\big\|_{L^4}|\tilde{v}|_2^{\frac{1}{2}}\left[\int_{\Omega}|\tilde{v}|\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}$$

$$+c\big\|\overline{|qT|}\big\|_{L^4}|\tilde{v}|_2^{\frac{1}{2}}\left[\int_{\Omega}|\tilde{v}|\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}. \tag{5.38}$$

By the Minkowski inequality, the Hölder inequality and (4.13), we have

$$\left[\int\limits_{S^2}\left(\int\limits_0^1 |\tilde{v}|^3\,d\xi\right)^2\right]^{\frac{1}{2}} \leqslant \int\limits_0^1\left[\int\limits_{S^2}\left(|\tilde{v}|^{\frac{3}{2}}\right)^4\right]^{\frac{1}{2}}d\xi$$
$$\leqslant c|\tilde{v}|_3^{\frac{3}{2}}\left[\int\limits_0^1\left(\left\|\nabla|\tilde{v}|^{\frac{3}{2}}\right\|_{L^2}^2+\left\||\tilde{v}|^{\frac{3}{2}}\right\|_{L^2}^2\right)d\xi\right]^{\frac{1}{2}}. \tag{5.39}$$

By the Minkowski inequality, the Hölder inequality and $\|u\|_{L^5}\leqslant c\|u\|_{L^3}^{\frac{3}{5}}\|u\|_{H^1}^{\frac{2}{5}}$, for any $u\in H^1(S^2)$, we get

$$\int\limits_{S^2}\left(\int\limits_0^1|\tilde{v}|^2\,d\xi\right)^{\frac{5}{2}} \leqslant \left[\int\limits_0^1\left(\int\limits_{S^2}|\tilde{v}|^5\right)^{\frac{2}{5}}d\xi\right]^{\frac{5}{2}} \leqslant c\|\tilde{v}\|^2|\tilde{v}|_3^3. \tag{5.40}$$

By the Minkowski inequality, (4.13), (4.15) and the Hölder inequality, we get

$$\left\|\overline{|T|}\right\|_{L^4}=\left[\int\limits_{S^2}\left(\int\limits_0^1|T|\,d\xi\right)^4\right]^{\frac{1}{4}}\leqslant|T|_4, \tag{5.41}$$

$$\left\|\overline{|qT|}\right\|_{L^4}=\left[\int\limits_{S^2}\left(\int\limits_0^1|qT|\,d\xi\right)^4\right]^{\frac{1}{4}}\leqslant c|q|_4^{\frac{1}{2}}\|q\|^{\frac{1}{2}}|T|_4^{\frac{1}{2}}\|T\|^{\frac{1}{2}}. \tag{5.42}$$

By the Young inequality, (4.13), (5.17), (5.29) we obtain from (5.38)–(5.42)

$$\frac{d|\tilde{v}|_3^3}{dt}+\int\limits_\Omega\left[\left(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\right)|\tilde{v}|+\frac{4}{9}\left|\nabla_{e_\theta}|\tilde{v}|^{\frac{3}{2}}\right|^2+\frac{4}{9}\left|\nabla_{e_\varphi}|\tilde{v}|^{\frac{3}{2}}\right|^2+|\tilde{v}|^3\right]$$
$$+\int\limits_\Omega\left(|\tilde{v}_\xi|^2|\tilde{v}|+\frac{4}{9}\left|\partial_\xi|\tilde{v}|^{\frac{3}{2}}\right|^2\right)$$
$$\leqslant c\left(\|\bar{v}\|_{L^2}^2\|\bar{v}\|_{H^1}^2+\|\tilde{v}\|^2\right)|\tilde{v}|_3^3+c\|T\|^2+c\left(1+\|q\|^2\right)|\tilde{v}|_2^2+c. \tag{5.43}$$

By the Uniform Gronwall Lemma, (5.9)–(5.11) and $|\tilde{v}|_3^3\leqslant|\tilde{v}|_2^{\frac{3}{2}}\|\tilde{v}\|_2^{\frac{3}{2}}$, we obtain

$$\left|\tilde{v}(t+3r)\right|_3^3\leqslant E_5, \tag{5.44}$$

where $E_5=E_5(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$ and $t\geqslant 0$.

5.6. L^4 estimates of $\tilde{v}$

Taking the inner product of Eq. (2.23) with $|\tilde{v}|^2\tilde{v}$ in L^2, similarly to (5.37), we obtain

$$\frac{1}{4}\frac{d|\tilde{v}|_4^4}{dt}+\int\limits_\Omega\left[\left(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\right)|\tilde{v}|^2+\frac{1}{2}\left|\nabla_{e_\theta}|\tilde{v}|^2\right|^2+\frac{1}{2}\left|\nabla_{e_\varphi}|\tilde{v}|^2\right|^2+|\tilde{v}|^4\right]$$

$$
\begin{aligned}
&+\int_\Omega \left(|\tilde{v}_\xi|^2|\tilde{v}|^2+\frac{1}{2}\big|\partial_\xi|\tilde{v}|^2\big|^2\right)\\
&=\int_\Omega \big[\bar{v}\cdot\nabla_{\tilde{v}}\big(|\tilde{v}|^2\tilde{v}\big)+|\tilde{v}|^2\tilde{v}\cdot\bar{v}\,\mathrm{div}\,\tilde{v}\big]\\
&+\int_\Omega\left[\left(\int_0^1\tilde{v}_\theta\,\tilde{v}\,d\xi\right)\cdot\nabla_{e_\theta}\big(|\tilde{v}|^2\tilde{v}\big)+\left(\int_0^1\tilde{v}_\varphi\,\tilde{v}\,d\xi\right)\cdot\nabla_{e_\varphi}\big(|\tilde{v}|^2\tilde{v}\big)\right]\\
&+\int_\Omega\left[\int_\xi^1\frac{bP}{p}(1+aq)T\,d\xi'-\int_0^1\int_\xi^1\frac{bP}{p}(1+aq)T\,d\xi'\,d\xi\right]\mathrm{div}\big(|\tilde{v}|^2\tilde{v}\big).
\end{aligned}
\tag{5.45}
$$

Similarly to (5.38), by the Hölder inequality, we derive from (5.45)

$$
\begin{aligned}
&\frac{1}{4}\frac{d|\tilde{v}|_4^4}{dt}+\int_\Omega\left[\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)|\tilde{v}|^2+\frac{1}{2}\big|\nabla_{e_\theta}|\tilde{v}|^2\big|^2+\frac{1}{2}\big|\nabla_{e_\varphi}|\tilde{v}|^2\big|^2+|\tilde{v}|^4\right]\\
&+\int_\Omega\big(|\tilde{v}_\xi|^2|\tilde{v}|^2+\big|\partial_\xi|\tilde{v}|^2\big|^2\big)\\
&\leqslant c\|\bar{v}\|_{L^4}\left[\int_\Omega|\tilde{v}|^2\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}\left[\int_{S^2}\left(\int_0^1|\tilde{v}|^4\,d\xi\right)^2\right]^{\frac{1}{4}}\\
&+c\left[\int_\Omega|\tilde{v}|^2\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}\left[\int_{S^2}\left(\int_0^1|\tilde{v}|^2\,d\xi\right)^3\right]^{\frac{1}{2}}\\
&+c\big\|\overline{|T|}\big\|_{L^4}\left[\int_\Omega|\tilde{v}|^2\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}\left[\int_{S^2}\left(\int_0^1|\tilde{v}|^2\,d\xi\right)^2\right]^{\frac{1}{4}}\\
&+c\big\|\overline{|qT|}\big\|_{L^4}\left[\int_\Omega|\tilde{v}|^2\big(|\nabla_{e_\theta}\tilde{v}|^2+|\nabla_{e_\varphi}\tilde{v}|^2\big)\right]^{\frac{1}{2}}\left[\int_{S^2}\left(\int_0^1|\tilde{v}|^2\,d\xi\right)^2\right]^{\frac{1}{4}}.
\end{aligned}
\tag{5.46}
$$

By the Minkowski inequality, the Hölder inequality and (4.13), we have

$$
\begin{aligned}
\left[\int_{S^2}\left(\int_0^1|\tilde{v}|^4\,d\xi\right)^2\right]^{\frac{1}{2}}&\leqslant\int_0^1\left[\int_{S^2}\big(|\tilde{v}|^2\big)^4\right]^{\frac{1}{2}}d\xi\\
&\leqslant c|\tilde{v}|_4^2\left(\int_0^1\big(\big\|\nabla|\tilde{v}|^2\big\|_{L^2}^2+\big\||\tilde{v}|^2\big\|_{L^2}^2\big)\,d\xi\right)^{\frac{1}{2}}.
\end{aligned}
\tag{5.47}
$$

By the Minkowski inequality, the Hölder inequality and (4.14), we get

$$\int_{S^2}\left(\int_0^1 |\tilde v|^2\,d\xi\right)^3 \leqslant \left[\int_0^1\left(\int_{S^2}|\tilde v|^6\right)^{\frac13} d\xi\right]^3 \leqslant c\|\tilde v\|^2|\tilde v|_4^4. \tag{5.48}$$

By the Young inequality, (4.13), (5.17), (5.29), (5.30), (5.41) and (5.42), we obtain from (5.46)–(5.48)

$$\begin{aligned}
&\frac{d|\tilde v|_4^4}{dt}+\int_\Omega\left[\left(|\nabla_{e_\theta}\tilde v|^2+|\nabla_{e_\varphi}\tilde v|^2\right)|\tilde v|^2+\frac12\left|\nabla_{e_\theta}|\tilde v|^2\right|^2+\frac12\left|\nabla_{e_\varphi}|\tilde v|^2\right|^2+|\tilde v|^4\right]\\
&\quad+\int_\Omega\left(|\tilde v_\xi|^2|\tilde v|^2+\frac12\left|\partial_\xi|\tilde v|^2\right|^2\right)\\
&\leqslant c\left(1+\|\bar v\|_{L^2}^2\|\bar v\|_{H^1}^2+\|\tilde v\|^2+\|q\|^2\right)|\tilde v|_4^4++c\|T\|^2+c.
\end{aligned}\tag{5.49}$$

By the Uniform Gronwall Lemma, (5.9)–(5.11), (5.44), (5.49) and $|\tilde v|_4^4\leqslant|\tilde v|_3^2\|\tilde v\|^2$, we obtain

$$\left|\tilde v(t+4r)\right|_4^4\leqslant E_6, \tag{5.50}$$

where $E_6=E_6(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$ and $t\geqslant 0$. From (5.49) and (5.50), we have

$$\begin{aligned}
&\int_{t+4r}^{t+5r}\left\{\int_\Omega\left[\left(|\nabla_{e_\theta}\tilde v|^2+|\nabla_{e_\varphi}\tilde v|^2\right)|\tilde v|^2+\frac12\left|\nabla_{e_\theta}|\tilde v|^2\right|^2+\frac12\left|\nabla_{e_\varphi}|\tilde v|^2\right|^2+|\tilde v|^4\right]\right.\\
&\quad\left.+\int_\Omega\left(|\tilde v_\xi|^2|\tilde v|^2+\frac12\left|\partial_\xi|\tilde v|^2\right|^2\right)\right\}\leqslant E_6^2+E_6=E_7.
\end{aligned}\tag{5.51}$$

By the Gronwall inequality, from (5.49) we obtain

$$\left|\tilde v(t)\right|_4^4\leqslant C_2, \tag{5.52}$$

where $C_2=C_2(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$ and $0\leqslant t<4r$.

5.7. H^1 estimates of $\bar v$

Taking the inner product of Eq. (2.22) with $-\Delta\bar v$ in L^2, we get

$$\begin{aligned}
&\frac12\frac{d\|\bar v\|_{H^1}^2}{dt}+\|\Delta\bar v\|_{L^2}^2\\
&=\int_{S^2}\left[\nabla_{\bar v}\bar v+\int_0^1(\tilde v\,\mathrm{div}\,\tilde v+\nabla_{\tilde v}\tilde v)\,d\xi\right]\cdot\Delta\bar v\\
&\quad+\int_{S^2}\left\{\left(\mathrm{grad}\,\Phi_s+\frac{f}{R_0}k\times\bar v\right)+\int_0^1\int_\xi^1\frac{bP}{p}\,\mathrm{grad}\big[(1+aq)T\big]\,d\xi'\,d\xi\right\}\cdot\Delta\bar v.
\end{aligned}\tag{5.53}$$

By the Hölder inequality, (4.13) and the Young inequality, we have

$$\left|\int_{S^2} (\nabla_{\bar{v}}\bar{v} \cdot \Delta\bar{v})\right| \leqslant c\|\bar{v}\|_{L^4}\left[\int_{S^2}\left(|\nabla_{e_\theta}\bar{v}|^2 + |\nabla_{e_\varphi}\bar{v}|^2\right)^2\right]^{\frac{1}{4}}\|\Delta\bar{v}\|_{L^2}$$
$$\leqslant c\|\bar{v}\|_{L^2}^{\frac{1}{2}}\|\bar{v}\|_{H^1}^{\frac{1}{2}}\left[\int_{S^2}\left(|\nabla_{e_\theta}\bar{v}|^2 + |\nabla_{e_\varphi}\bar{v}|^2\right)\right]^{\frac{1}{4}}$$
$$\times\left\{\left[\int_{S^2}\left(|\nabla_{e_\theta}\bar{v}|^2 + |\nabla_{e_\varphi}\bar{v}|^2\right)\right]^{\frac{1}{4}} + \|\Delta\bar{v}\|_{L^2}^{\frac{1}{2}}\right\}\|\Delta\bar{v}\|_{L^2(S^2)}$$
$$\leqslant c\left(\|\bar{v}\|_{L^2}^2 + \|\bar{v}\|_{H^1}^2 + \|\bar{v}\|_{L^2}^2\|\bar{v}\|_{H^1}^2\right)\|\bar{v}\|_{H^1}^2 + \varepsilon\|\Delta\bar{v}\|_{L^2}^2. \tag{5.54}$$

By the Hölder inequality and the Minkowski inequality, we obtain

$$\left|\int_{S^2}\left(\int_0^1 (\tilde{v}\operatorname{div}\tilde{v} + \nabla_{\tilde{v}}\tilde{v})\,d\xi \cdot \Delta\bar{v}\right)\right| \leqslant c\int_\Omega |\tilde{v}|^2\left(|\nabla_{e_\theta}\tilde{v}|^2 + |\nabla_{e_\varphi}\tilde{v}|^2\right) + \varepsilon\|\Delta\bar{v}\|_{L^2}^2. \tag{5.55}$$

From (5.53)–(5.55), by $(\frac{f}{R_0}k \times \bar{v}) \cdot \Delta\bar{v} = 0$ and Lemma 4.1, choosing ε small enough, we obtain

$$\frac{d\|\bar{v}\|_{H^1}^2}{dt} + \|\Delta\bar{v}\|_{L^2}^2 \leqslant c\left(\|\bar{v}\|_{H^1}^2 + \|\bar{v}\|_{L^2}^2\|\bar{v}\|_{H^1}^2\right)\|\bar{v}\|_{H^1}^2 + c\int_\Omega |\tilde{v}|^2\left(|\nabla_{e_\theta}\tilde{v}|^2 + |\nabla_{e_\varphi}\tilde{v}|^2\right). \tag{5.56}$$

By the Uniform Gronwall Lemma, (5.9)–(5.11) and (5.51), we get

$$\left\|\bar{v}(t+5r)\right\|_{H^1}^2 \leqslant E_8, \tag{5.57}$$

where $E_8 = E_8(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1) > 0$. By the Gronwall inequality, from (5.56) we obtain

$$\left\|\bar{v}(t)\right\|_{H^1}^2 \leqslant C_3, \tag{5.58}$$

where $C_3 = C_3(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1) > 0$ and $0 \leqslant t < 5r$.

5.8. L^2 estimates of v_ξ

Taking the derivative, with respect to ξ, of Eq. (2.11), then taking the inner product of equation obtained with v_ξ in L^2, we obtain

$$\frac{1}{2}\frac{d|v_\xi|_2^2}{dt} + \int_\Omega\left(|\nabla_{e_\theta}v_\xi|^2 + |\nabla_{e_\varphi}v_\xi|^2 + |v_\xi|^2\right) + \int_\Omega\left|\frac{\partial v_\xi}{\partial\xi}\right|^2$$
$$= -\int_\Omega\left[\nabla_v v_\xi + \left(\int_\xi^1 \operatorname{div} v\,d\xi'\right)\frac{\partial v_\xi}{\partial\xi}\right]\cdot v_\xi - \int_\Omega\left[\nabla_{v_\xi}v - (\operatorname{div} v)\frac{\partial v}{\partial\xi}\right]\cdot v_\xi$$
$$-\int_\Omega\left(\frac{f}{R_0}k \times v_\xi\right)\cdot v_\xi + \int_\Omega \frac{bP}{p}\operatorname{grad}\left[(1+aq)T\right]\cdot v_\xi. \tag{5.59}$$

By integration by parts, the Hölder inequality, (4.16) and the Young inequality, we have

$$-\int_\Omega \left[\nabla_{v_\xi} v - (\operatorname{div} v)\frac{\partial v}{\partial \xi}\right] \cdot v_\xi \leqslant c\int_\Omega |v||v_\xi|\big(|\nabla_{e_\theta} v_\xi|^2 + |\nabla_{e_\varphi} v_\xi|^2\big)^{\frac{1}{2}}$$

$$\leqslant c|v|_4|v_\xi|_2^{\frac{1}{4}}\|v_\xi\|^{\frac{3}{4}}\left[\int_\Omega \big(|\nabla_{e_\theta} v_\xi|^2 + |\nabla_{e_\varphi} v_\xi|^2\big)\right]^{\frac{1}{2}}$$

$$\leqslant \varepsilon\|v_\xi\|^2 + c|v|_4^8|v_\xi|_2^2. \tag{5.60}$$

By Lemma 4.1, the Hölder inequality and the Young inequality, we obtain

$$\int_\Omega \frac{bP}{p}\operatorname{grad}\big[(1+aq)T\big] \cdot v_\xi = -\int_\Omega \frac{bP}{p}(1+aq)T \operatorname{div} v_\xi$$

$$\leqslant c|T|_2^2 + c|q|_4^2|T|_4^2 + \varepsilon\|v_\xi\|^2. \tag{5.61}$$

By integration by parts, Lemma 4.2 and $(\frac{f}{R_0}k \times v_\xi) \cdot v_\xi = 0$, choosing ε small enough, we derive from (5.59)–(5.61),

$$\frac{d|v_\xi|_2^2}{dt} + \int_\Omega \big(|\nabla_{e_\theta} v_\xi|^2 + |\nabla_{e_\varphi} v_\xi|^2 + |v_\xi|^2\big) + \int_\Omega \left|\frac{\partial v_\xi}{\partial \xi}\right|^2$$

$$\leqslant c\big(|\bar{v}|_{H^1}^8 + |\tilde{v}|_4^8\big)|v_\xi|_2^2 + c|T|_2^2 + c|q|_4^4 + c|T|_4^4. \tag{5.62}$$

By the Uniform Gronwall Lemma, (5.10), (5.17), (5.29), (5.50), (5.57) and (5.62), we get

$$\big|v_\xi(t+6r)\big|_2^2 \leqslant E_9, \tag{5.63}$$

where $E_9 = E_9(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1) > 0$ and $t \geqslant 0$. From (5.62) and (5.63), we have

$$c_1\int_{t+6r}^{t+7r} \|v_\xi\|^2 \leqslant E_9^2 + E_9 = E_{10}. \tag{5.64}$$

By the Gronwall inequality, from (5.62), we obtain

$$\big|v_\xi(t)\big|_2^2 \leqslant C_4, \tag{5.65}$$

where $C_4 = C_4(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1) > 0$ and $0 \leqslant t < 6r$.

5.9. L^2 estimates of T_ξ, q_ξ

Taking the derivative, with respect to ξ, of Eqs. (2.12), then taking the inner product of equation obtained with T_ξ in $L^2(\Omega)$, we obtain

$$\frac{1}{2}\frac{d|T_\xi|_2^2}{dt}+\int_\Omega|\nabla T_\xi|^2+\int_\Omega|T_{\xi\xi}|^2-\int_{S^2}(T_\xi|_{\xi=1}\cdot T_{\xi\xi}|_{\xi=1})$$
$$=-\int_\Omega\left[\nabla_v T_\xi+W(v)\frac{\partial T_\xi}{\partial\xi}\right]T_\xi-\int_\Omega\left[\nabla_{v_\xi}T-(\operatorname{div}v)\frac{\partial T}{\partial\xi}\right]T_\xi+\int_\Omega Q_{1\xi}T_\xi$$
$$+\int_\Omega\frac{bP}{p}\left[-(1+aq)(\operatorname{div}v)-\frac{(P-p_0)}{p}(1+aq)W(v)+aq_\xi W(v)\right]T_\xi. \tag{5.66}$$

By integration by parts, the Hölder inequality, (4.16), the Poincaré inequality and the Young inequality, we obtain

$$\left|\int_\Omega\left[\nabla_{v_\xi}T-\operatorname{div}(v)\frac{\partial T}{\partial\xi}\right]T_\xi\right|$$
$$\leqslant c\int_\Omega\left[\left(|\nabla_{e_\theta}v_\xi|^2+|\nabla_{e_\varphi}v_\xi|^2\right)^{\frac{1}{2}}|T||T_\xi|+|v_\xi||T||\nabla T_\xi|+|v||\nabla T_\xi||T_\xi|\right]$$
$$\leqslant c\int_\Omega\left(|\nabla_{e_\theta}v_\xi|^2+|\nabla_{e_\varphi}v_\xi|^2\right)+\frac{\varepsilon}{2}|\nabla T_\xi|_2^2+c|T|_4^2|T_\xi|_4^2+c|v_\xi|_4^2|T|_4^2+c|v|_4^2|T_\xi|_4^2$$
$$\leqslant\varepsilon\left(|T_{\xi\xi}|_2^2+|\nabla T_\xi|_2^2\right)+c\left[|v_{\xi\xi}|_2^2+\int_\Omega\left(|\nabla_{e_\theta}v_\xi|^2+|\nabla_{e_\varphi}v_\xi|^2\right)\right]$$
$$+c|T|_4^8|v_\xi|_2^2+c\left(|T|_4^8+|v|_4^8\right)|T_\xi|_2^2. \tag{5.67}$$

By integrating by parts, the Hölder inequality, the Minkowski inequality, the Poincaré inequality, the Young inequality and (4.16), we obtain

$$\left|\int_\Omega\left[-\frac{bP}{p}(1+aq)(\operatorname{div}v)-\frac{bP(P-p_0)}{p^2}(1+aq)W(v)+\frac{abP}{p}q_\xi W(v)\right]T_\xi\right|$$
$$\leqslant\left|\int_\Omega\left[\frac{bP}{p}\nabla q\cdot vT_\xi+\frac{bP}{p}(1+aq)v\cdot\nabla T_\xi\right]\right|+\left|\int_\Omega\left[\frac{bP(P-p_0)}{p^2}(1+aq)W(v)\right]T_\xi\right|$$
$$+\left|\int_\Omega\left[\frac{abP}{p}(\nabla q_\xi)\left(\int_\xi^1 v\right)T_\xi+\frac{abP}{p}q_\xi\left(\int_\xi^1 v\right)\nabla T_\xi\right]\right|$$
$$\leqslant\varepsilon\left(|\nabla q_\xi|_2^2+|\nabla T_\xi|_2^2\right)+c|\nabla q|_2^2+c\|v\|^2+c\left(|v|_4^2+|q|_4^2\right)|T_\xi|_4^2$$
$$+c|v|_4^2\left(|q_\xi|_4^2+|q|_4^2\right)+c|T_\xi|_2^2$$
$$\leqslant\varepsilon\left(|\nabla T_\xi|_2^2+|T_{\xi\xi}|_2^2\right)+\varepsilon\left(|\nabla q_\xi|_2^2+|q_{\xi\xi}|_2^2\right)+c\left(|v|_4^8+|q|_4^8+1\right)|T_\xi|_2^2$$
$$+c|v|_4^8|q_\xi|_2^2+c\left(|\nabla q|_2^2+\|v\|^2\right)+c|v|_4^2|q|_4^2. \tag{5.68}$$

From (2.15), by taking the trace on $\xi=1$ of Eq. (2.12), we get

$$
\begin{aligned}
&-\int_{S^2} (T_\xi|_{\xi=1} T_{\xi\xi}|_{\xi=1}) \\
&= \alpha_s \int_{S^2} T|_{\xi=1}\left[\frac{\partial T|_{\xi=1}}{\partial t} + (\nabla_v T)|_{\xi=1} - \Delta T|_{\xi=1} - Q_1|_{\xi=1}\right] \\
&= \alpha_s\left(\frac{1}{2}\frac{d|T|_{\xi=1}|_2^2}{dt} + |\nabla T|_{\xi=1}|_2^2\right) + \alpha_s \int_{S^2} T|_{\xi=1}\big[(\nabla_v T)|_{\xi=1} - Q_1|_{\xi=1}\big].
\end{aligned}
\tag{5.69}
$$

By Lemma 4.2, we have

$$
\begin{aligned}
&-\alpha_s \int_{S^2} T|_{\xi=1}\big[(\nabla_v T)|_{\xi=1} - Q_1|_{\xi=1}\big] \\
&= -\frac{\alpha_s}{2}\int_{S^2} \big(\nabla_v T^2\big)\big|_{\xi=1} + \alpha_s \int_{S^2} T|_{\xi=1} Q_1|_{\xi=1} \\
&= \frac{\alpha_s}{2}\int_{S^2} T^2|_{\xi=1} \operatorname{div} v|_{\xi=1} + \alpha_s \int_{S^2} T|_{\xi=1} Q_1|_{\xi=1} \\
&= \frac{\alpha_s}{2}\int_{S^2} T^2|_{\xi=1}\left(\int_\xi^1 \operatorname{div} v_\xi \, d\xi' + \operatorname{div} v\right) + \alpha_s \int_{S^2} T|_{\xi=1} Q_1|_{\xi=1} \\
&\leqslant c|T|_{\xi=1}|_4^4 + c\|v_\xi\|^2 + c\|v\|^2 + c|T|_{\xi=1}|_2^2 + c|Q_1|_{\xi=1}|_2^2.
\end{aligned}
\tag{5.70}
$$

By Lemma 4.3, the Young inequality, (5.17), (5.29), (5.30), (5.50), (5.52), (5.57), (5.58), (5.63) and (5.65), we derive from (5.66)–(5.70)

$$
\begin{aligned}
&\frac{1}{2}\frac{d(|T_\xi|_2^2 + \alpha_s|T|_{\xi=1}|_2^2)}{dt} + \int_\Omega |\nabla T_\xi|^2 + \int_\Omega |T_{\xi\xi}|^2 + \alpha_s|\nabla T|_{\xi=1}|_2^2 \\
&\leqslant 2\varepsilon\big(|T_{\xi\xi}|_2^2 + |\nabla T_\xi|_2^2\big) + \varepsilon\big(|\nabla q_\xi|_2^2 + |q_{\xi\xi}|_2^2\big) + c|T_\xi|_2^2 + c|q_\xi|_2^2 + c\|v_\xi\|^2 + c\|v\|^2 \\
&\quad + c|\nabla q|_2^2 + c|T|_{\xi=1}|_4^4 + c|T|_{\xi=1}|_2^2 + c|Q_1|_{\xi=1}|_2^2 + c|Q_1|_2^2 + c|Q_{1\xi}|_2^2 + c.
\end{aligned}
\tag{5.71}
$$

Similarly to (5.71), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d(|q_\xi|_2^2 + \beta_s|q|_{\xi=1}|_2^2)}{dt} + \int_\Omega |\nabla q_\xi|^2 + \int_\Omega |q_{\xi\xi}|^2 + \beta_s|\nabla q|_{\xi=1}|_2^2 \\
&\leqslant \varepsilon\big(|\nabla q_\xi|_2^2 + |q_{\xi\xi}|_2^2\big) + c|q_\xi|_2^2 + c\|v_\xi\|^2 + c\|v\|^2 + c|v_\xi|_2^2 \\
&\quad + c|q|_{\xi=1}|_4^4 + c|q|_{\xi=1}|_2^2 + c|Q_2|_{\xi=1}|_2^2 + c|Q_2|_2^2 + c|Q_{2\xi}|_2^2.
\end{aligned}
\tag{5.72}
$$

From (5.71) and (5.72), choosing ε small enough, we obtain

$$\frac{d(|T_\xi|_2^2+|q_\xi|_2^2+\beta_s|q|_{\xi=1}|_2^2+\alpha_s|T|_{\xi=1}|_2^2)}{dt}+\int_\Omega|\nabla T_\xi|^2+\int_\Omega|T_{\xi\xi}|^2$$
$$+\alpha_s|\nabla T|_{\xi=1}|_2^2+\int_\Omega|\nabla q_\xi|^2+\int_\Omega|q_{\xi\xi}|^2+\beta_s|\nabla q|_{\xi=1}|_2^2$$
$$\leqslant c+c\big(|T_\xi|_2^2+|q_\xi|_2^2\big)+c\|v_\xi\|^2+c\|v\|^2+c\|q\|^2+c|T|_{\xi=1}|_4^4+c|T|_{\xi=1}|_2^2$$
$$+c|q|_{\xi=1}|_4^4+c|q|_{\xi=1}|_2^2+c\big(|Q_1|_{\xi=1}|_2^2+|Q_2|_{\xi=1}|_2^2\big)+c\big(\|Q_1\|_1^2+\|Q_2\|_1^2\big). \tag{5.73}$$

By the Uniform Gronwall Lemma, (5.10), (5.18), (5.64) and (5.73), we get

$$\big|T_\xi(t+7r)\big|_2^2+\big|q_\xi(t+7r)\big|_2^2\leqslant E_{11}, \tag{5.74}$$

where $E_{11}=E_{11}(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$. From (5.73) and (5.74), we have

$$c_1\int_{t+7r}^{t+8r}\big(\|T_\xi\|^2+\|q_\xi\|^2+|\nabla T|_{\xi=1}|_2^2+|\nabla q|_{\xi=1}|_2^2\big)\leqslant E_{11}^2+2E_{11}+E_1=E_{12}. \tag{5.75}$$

By the Gronwall inequality, from (5.73) we obtain

$$\big|T_\xi(t)\big|_2^2+\big|q_\xi(t)\big|_2^2\leqslant C_5, \tag{5.76}$$

where $C_5=C_5(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$ and $0\leqslant t<7r$.

5.10. H^1 estimates of v, T, q

Taking the inner product of Eq. (2.11) with $-\Delta v$ in L^2, we get

$$\frac{1}{2}\frac{d\int_\Omega(|\nabla_{e_\theta}v|^2+|\nabla_{e_\varphi}v|^2+|v|^2)}{dt}+|\Delta v|_2^2+\int_\Omega\big(|\nabla_{e_\theta}v_\xi|^2+|\nabla_{e_\varphi}v_\xi|^2+|v_\xi|^2\big)$$
$$=\int_\Omega\big[\nabla_v v+W(v)v_\xi\big]\cdot\Delta v+\int_\Omega\left\{\int_\xi^1\frac{bP}{p}\,\mathrm{grad}\big[(1+aq)T\big]\,d\xi'\right\}\cdot\Delta v$$
$$+\int_\Omega\left(\frac{f}{R_0}k\times v+\mathrm{grad}\,\Phi_s\right)\cdot\Delta v. \tag{5.77}$$

By the Hölder inequality, (4.16) and the Young inequality, we have

$$\left|\int_\Omega\nabla_v v\cdot\Delta v\right|$$
$$\leqslant\int_\Omega|v|\big(|\nabla_{e_\theta}v|^2+|\nabla_{e_\varphi}v|^2\big)^{\frac{1}{2}}|\Delta v|$$

$$
\begin{aligned}
&\leqslant c|v|_4^2\bigg[\int\limits_\Omega \big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\big)\bigg]^{\frac14}\bigg[\int\limits_\Omega \big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2 \\
&\quad+|\nabla_{e_\theta} v_\xi|^2+|\nabla_{e_\varphi} v_\xi|^2+|\Delta v|^2\big)\bigg]^{\frac34}+\varepsilon|\Delta v|_2^2 \\
&\leqslant c\big(|v|_4^8+|v|_4^2\big)\int\limits_\Omega \big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\big)+2\varepsilon\bigg[|\Delta v|_2^2+\int\limits_\Omega \big(|\nabla_{e_\theta} v_\xi|^2+|\nabla_{e_\varphi} v_\xi|^2\big)\bigg]. \qquad (5.78)
\end{aligned}
$$

By the Hölder inequality, the Minkowski inequality, (4.13) and the Young inequality, we obtain

$$
\begin{aligned}
&\bigg|\int\limits_\Omega W(v)v_\xi\cdot\Delta v\bigg| \\
&\quad\leqslant \int\limits_{S^2}\bigg[\int\limits_0^1 \big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\big)^{\frac12}\,d\xi\int\limits_0^1 |v_\xi||\Delta v|\,d\xi\bigg] \\
&\quad\leqslant c\Bigg\{\int\limits_0^1\bigg[\int\limits_{S^2}\big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\big)^2\bigg]^{\frac12}d\xi\Bigg\}\int\limits_0^1\bigg(\int\limits_{S^2}|v_\xi|^4\bigg)^{\frac12}d\xi+\varepsilon|\Delta v|_2^2 \\
&\quad\leqslant c\Bigg\{\int\limits_0^1\bigg[\int\limits_{S^2}\big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\big)\bigg]^{\frac12}\cdot\bigg[\int\limits_{S^2}\big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2+|\Delta v|^2\big)\bigg]^{\frac12}d\xi\Bigg\} \\
&\qquad\cdot c\Bigg\{\int\limits_0^1\bigg(\int\limits_{S^2}|v_\xi|^2\bigg)^{\frac12}\cdot\bigg[\int\limits_{S^2}\big(|\nabla_{e_\theta} v_\xi|^2+|\nabla_{e_\varphi} v_\xi|^2+|v_\xi|^2\big)\bigg]^{\frac12}d\xi\Bigg\}+\varepsilon|\Delta v|_2^2 \\
&\quad\leqslant 2\varepsilon|\Delta v|_2^2+c\bigg[2|v_\xi|_2^2+|v_\xi|_2^4+\big(|v_\xi|_2^2+1\big)\int\limits_\Omega\big(|\nabla_{e_\theta} v_\xi|^2+|\nabla_{e_\varphi} v_\xi|^2\big)\bigg] \\
&\qquad\cdot\int\limits_\Omega\big(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\big). \qquad (5.79)
\end{aligned}
$$

By the Hölder inequality, the Young inequality, the Minkowski inequality, and (4.13), we have

$$
\begin{aligned}
&\bigg|\int\limits_\Omega\int\limits_\xi^1 \frac{bP}{p}\,\mathrm{grad}\big[(1+aq)T\big]\,d\xi'\cdot\Delta v\bigg| \\
&\quad\leqslant c\Bigg[\int\limits_\Omega\bigg(\int\limits_0^1|q|^2\,d\xi\bigg)^2\Bigg]^{\frac12}\Bigg[\int\limits_\Omega\bigg(\int\limits_0^1|\nabla T|^2\,d\xi\bigg)^2\Bigg]^{\frac12} \\
&\qquad+c\Bigg[\int\limits_\Omega\bigg(\int\limits_0^1|T|^2\,d\xi\bigg)^2\Bigg]^{\frac12}\Bigg[\int\limits_\Omega\bigg(\int\limits_0^1|\nabla q|^2\,d\xi\bigg)^2\Bigg]^{\frac12}+c|\nabla T|_2^2+\varepsilon|\Delta v|_2^2
\end{aligned}
$$

$$\leqslant c|q|_4^2 \int_0^1 \left[\|\nabla T\|_{L^2}\left(\|\nabla T\|_{L^2}^2 + \|\Delta T\|_{L^2}^2\right)^{\frac{1}{2}}\right] d\xi$$

$$+ c|T|_4^2 \int_0^1 \left[\|\nabla q\|_{L^2}\left(\|\nabla q\|_{L^2}^2 + \|\Delta q\|_{L^2}^2\right)^{\frac{1}{2}}\right] d\xi + c|\nabla T|_2^2 + \varepsilon|\Delta v|_2^2$$

$$\leqslant c|q|_4^2|\nabla T|_2^2 + c|T|_4^2|\nabla q|_2^2 + c|q|_4^4|\nabla T|_2^2 + c|T|_4^4|\nabla q|_2^2 + c|\nabla T|_2^2$$

$$+ \varepsilon|\Delta v|_2^2 + \varepsilon|\Delta T|_2^2 + \varepsilon|\Delta q|_2^2. \tag{5.80}$$

By Lemma 4.1, $(\frac{f}{R_0}k \times v) \cdot \Delta v = 0$, (5.17), (5.29), (5.30), (5.50), (5.52), (5.57), (5.58), (5.63) and (5.65), we derive from (5.77)–(5.80)

$$\frac{1}{2}\frac{d\int_\Omega(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2)}{dt} + |\Delta v|_2^2 + \int_\Omega \left(|\nabla_{e_\theta} v_\xi|^2 + |\nabla_{e_\varphi} v_\xi|^2 + |v_\xi|^2\right)$$

$$\leqslant c\left(1 + \|v_\xi\|^2\right)\int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\right) + c|\nabla T|_2^2 + c|\nabla q|_2^2$$

$$+ 2\varepsilon\int_\Omega \left(|\nabla_{e_\theta} v_\xi|^2 + |\nabla_{e_\varphi} v_\xi|^2\right) + 5\varepsilon|\Delta v|_2^2 + \varepsilon|\Delta T|_2^2 + \varepsilon|\Delta q|_2^2. \tag{5.81}$$

Similarly to (5.81), we obtain

$$\frac{1}{2}\frac{d|\nabla T|_2^2}{dt} + |\Delta T|_2^2 + |\nabla T_\xi|_2^2 + \alpha_s|\nabla T|_{\xi=1}|_2^2$$

$$\leqslant 5\varepsilon|\Delta T|_2^2 + 2\varepsilon|\nabla T_\xi|_2^2 + 2\varepsilon|\Delta v|_2^2 + c\left(1 + \int_\Omega |\nabla T_\xi|^2\right)\int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\right)$$

$$+ c|\nabla T|_2^2 + c|Q_1|_2^2, \tag{5.82}$$

$$\frac{1}{2}\frac{d|\nabla q|_2^2}{dt} + |\Delta q|_2^2 + |\nabla q_\xi|_2^2 + \beta_s|\nabla q|_{\xi=1}|_2^2$$

$$\leqslant 4\varepsilon|\Delta q|_2^2 + 2\varepsilon|\nabla q_\xi|_2^2 + \varepsilon|\Delta v|_2^2 + c\left(1 + \int_\Omega |\nabla q_\xi|^2\right)\int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\right)$$

$$+ c|\nabla q|_2^2 + c|Q_2|_2^2. \tag{5.83}$$

From (5.81)–(5.83), choosing ε small enough, we obtain

$$\frac{d[\int_\Omega(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2) + |\nabla T|_2^2 + |\nabla q|_2^2]}{dt}$$

$$+ |\Delta v|_2^2 + |\Delta T|_2^2 + |\Delta q|_2^2 + \int_\Omega \left(|\nabla_{e_\theta} v_\xi|^2 + |\nabla_{e_\varphi} v_\xi|^2 + |v_\xi|^2\right)$$

$$+ |\nabla T_\xi|_2^2 + \alpha_s|\nabla T|_{\xi=1}|_2^2 + |\nabla q_\xi|_2^2 + \beta_s|\nabla q|_{\xi=1}|_2^2$$

$$\leqslant c\left(1+\|v_\xi\|^2+\int_\Omega |\nabla T_\xi|^2+\int_\Omega |\nabla q_\xi|^2\right)$$
$$\cdot\left[\int_\Omega \left(|\nabla_{e_\theta} v|^2+|\nabla_{e_\varphi} v|^2\right)+|\nabla T|_2^2+|\nabla q|_2^2\right]+c|Q_1|_2^2+c|Q_2|_2^2. \tag{5.84}$$

By the Uniform Gronwall Lemma, (5.10), (5.64), (5.75) and (5.84), we get

$$\left|\nabla_{e_\theta} v(t+8r)\right|_2^2+\left|\nabla_{e_\varphi} v(t+8r)\right|_2^2+\left|\nabla T(t+8r)\right|_2^2+\left|\nabla q(t+8r)\right|_2^2 \leqslant E_{13}, \tag{5.85}$$

where $E_{13}=E_{13}(\|U_0\|,\|Q_1\|_1,\|Q_2\|_1)>0$. By the Gronwall inequality, from (5.84) we obtain

$$\left|\nabla_{e_\theta} v(t)\right|_2^2+\left|\nabla_{e_\varphi} v(t)\right|_2^2+\left|\nabla T(t)\right|_2^2+\left|\nabla q(t)\right|_2^2 \leqslant C_6, \tag{5.86}$$

where $C_6=C_6(\|U_0\|,\|Q_1\|,\|Q_2\|)>0$ and $0\leqslant t<8r$.

6. The existence and uniqueness of global strong solutions

6.1. The existence of global strong solutions

Proof of Proposition 3.1. By Proposition 5.3, we can use the method of contradiction to prove Proposition 3.1. Indeed, let U be a strong solution to the system (2.11)–(2.17) on the maximal interval $[0,\mathcal{T}_*]$. If $\mathcal{T}_*<+\infty$, then

$$\limsup_{t\to\mathcal{T}_*^-}\|U\|=+\infty,$$

which is impossible from (5.10), (5.63), (5.65), (5.74), (5.76), (5.85), (5.86). The proof is complete. □

6.2. The uniqueness of global strong solutions

Proof of Proposition 3.2. Let (v_1,T_1,q_1) and (v_2,T_2,q_2) be two strong solutions of (2.11)–(2.17) on the time interval $[0,\mathcal{T}]$ with corresponding geopotentials Φ_{s_1}, Φ_{s_2}, and initial data $((v_0)_1,(T_0)_1,(q_0)_1)$, $((v_0)_2,(T_0)_2,(q_0)_2)$, respectively. Define $v=v_1-v_2$, $T=T_1-T_2$, $q=q_1-q_2$, $\Phi_s=\Phi_{s_1}-\Phi_{s_2}$. Then v, T, q, Φ_s satisfy the following system

$$\frac{\partial v}{\partial t}-\Delta v-\frac{\partial^2 v}{\partial \xi^2}+\nabla_{v_1}v+\nabla_v v_2+W(v_1)\frac{\partial v}{\partial \xi}+W(v)\frac{\partial v_2}{\partial \xi}+\frac{f}{R_0}k\times v+\operatorname{grad}\Phi_s$$
$$+\int_\xi^1 \frac{bP}{p}\operatorname{grad}T\,d\xi'+\int_\xi^1\frac{abP}{p}\operatorname{grad}(q_1T)\,d\xi'+\int_\xi^1\frac{abP}{p}\operatorname{grad}(qT_2)\,d\xi'=0, \tag{6.1}$$

$$\frac{\partial T}{\partial t}-\Delta T-\frac{\partial^2 T}{\partial \xi^2}+\nabla_{v_1}T+\nabla_v T_2+W(v_1)\frac{\partial T}{\partial \xi}+W(v)\frac{\partial T_2}{\partial \xi}-\frac{bP}{p}W(v)$$
$$-\frac{abP}{p}q_1W(v)-\frac{abP}{p}qW(v_2)=0, \tag{6.2}$$

$$\frac{\partial q}{\partial t}-\Delta q-\frac{\partial^2 q}{\partial \xi^2}+\nabla_{v_1}q+\nabla_v q_2+W(v_1)\frac{\partial q}{\partial \xi}+W(v)\frac{\partial q_2}{\partial \xi}=0, \tag{6.3}$$

$$(v|_{t=0}, T|_{t=0}, q|_{t=0}) = \big((v_0)_1 - (v_0)_2, (T_0)_1 - (T_0)_2, (q_0)_1 - (q_0)_2\big), \tag{6.4}$$

$$\xi = 1: \quad \frac{\partial v}{\partial \xi} = 0, \quad \frac{\partial T}{\partial \xi} = -\alpha_s T, \quad \frac{\partial q}{\partial \xi} = -\beta_s q, \tag{6.5}$$

$$\xi = 0: \quad \frac{\partial v}{\partial \xi} = 0, \quad \frac{\partial T}{\partial \xi} = 0, \quad \frac{\partial q}{\partial \xi} = 0. \tag{6.6}$$

We take the inner product of Eq. (6.1) with v in L^2 and obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d|v|_2^2}{dt} + \int_\Omega \big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2\big) + \int_\Omega |v_\xi|^2 \\
&= -\int_\Omega \left[\nabla_{v_1} v + W(v_1)\frac{\partial v}{\partial \xi}\right]\cdot v - \int_\Omega v\cdot\nabla_v v_2 - \int_\Omega W(v)\frac{\partial v_2}{\partial \xi}\cdot v \\
&\quad - \int_\Omega \left(\frac{f}{R_0} k\times v + \operatorname{grad}\Phi_s\right)\cdot v - \int_\Omega \left(\int_\xi^1 \frac{bP}{p}\operatorname{grad} T\, d\xi'\right)\cdot v \\
&\quad - \int_\Omega \left[\int_\xi^1 \frac{abP}{p}\operatorname{grad}(q_1 T)\, d\xi'\right]\cdot v - \int_\Omega \left[\int_\xi^1 \frac{abP}{p}\operatorname{grad}(q T_2)\, d\xi'\right]\cdot v.
\end{aligned} \tag{6.7}$$

Using Lemma 4.2, the Hölder inequality, the Young inequality and (4.16), we get

$$\begin{aligned}
\left|\int_\Omega v\cdot\nabla_v v_2\right| &= \left|\int_\Omega (v_2\cdot\nabla_v v + v_2\cdot v \operatorname{div} v)\right| \\
&\leqslant c\int_\Omega |v||v_2|\big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\big)^{\frac{1}{2}} \\
&\leqslant \varepsilon\int_\Omega \big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\big) + c|v|_2^{\frac{1}{2}}|v_2|_4^2\|v\|^{\frac{3}{2}} \\
&\leqslant 2\varepsilon\|v\|^2 + c|v_2|_4^8|v|_2^2.
\end{aligned} \tag{6.8}$$

By the Hölder inequality, the Young inequality, the Minkowski inequality and (4.13), we obtain

$$\begin{aligned}
&\left|\int_\Omega W(v)\frac{\partial v_2}{\partial \xi}\cdot v\right| \\
&\leqslant \varepsilon\int_\Omega \big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\big) + c\left[\int_0^1\left(\int_{S^2}|v_{2\xi}|^4\right)^{\frac{1}{2}} d\xi\right]\int_0^1\left(\int_{S^2}|v|^4\right)^{\frac{1}{2}} d\xi \\
&\leqslant \varepsilon\int_\Omega \big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2\big) + c\int_0^1\left[\|v_{2\xi}\|_{L^2}\left(\int_{S^2}\big(|\nabla_{e_\theta} v_{2\xi}|^2 + |\nabla_{e_\varphi} v_{2\xi}|^2\big)\right)^{\frac{1}{2}}\right] d\xi
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^1 \left[\|v\|_{L^2} \left(\int_{S^2} \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2 \right) \right)^{\frac{1}{2}} \right] d\xi \\
&\leqslant 2\varepsilon \|v\|^2 + c\left[\left(|v_{2\xi}|_2^2 + 1 \right) \int_\Omega \left(|\nabla_{e_\theta} v_{2\xi}|^2 + |\nabla_{e_\varphi} v_{2\xi}|^2 \right) + |v_{2\xi}|_2^2 \right] |v|_2^2. \qquad (6.9)
\end{aligned}
$$

By Lemma 4.1, the Hölder inequality, the Young inequality, the Minkowski inequality and (4.13), we have

$$
\begin{aligned}
&\left| \int_\Omega \left[\int_\xi^1 \frac{abP}{p} \operatorname{grad}(qT_2)\, d\xi' \right] \cdot v \right| \\
&\quad \leqslant c\left[\int_0^1 \left(\int_{S^2} |q|^4 \right)^{\frac{1}{2}} d\xi \right] \int_0^1 \left(\int_{S^2} |T_2|^4 \right)^{\frac{1}{2}} d\xi + \varepsilon \int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 \right) \\
&\quad \leqslant c\left(|T_2|_4^2 + |T_2|_4^4 \right) |q|_2^2 + \varepsilon |\nabla q|_2^2 + \varepsilon \int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 \right). \qquad (6.10)
\end{aligned}
$$

By Lemma 4.1, Lemma 4.3 and $(\frac{f}{R_0} k \times v) \cdot v = 0$, we derive from (6.7)–(6.10)

$$
\begin{aligned}
&\frac{1}{2} \frac{d|v|_2^2}{dt} + \int_\Omega \left(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2 \right) + \int_\Omega |v_\xi|^2 \\
&\quad \leqslant 5\varepsilon \|v\|^2 + \varepsilon |\nabla q|_2^2 + c\left(|T_2|_4^2 + |T_2|_4^4 \right) |q|_2^2 \\
&\qquad + c\left[|v_2|_4^8 + |v_{2\xi}|_2^2 + \left(|v_{2\xi}|_2^2 + 1 \right) \int_\Omega \left(|\nabla_{e_\theta} v_{2\xi}|^2 + |\nabla_{e_\varphi} v_{2\xi}|^2 \right) \right] |v|_2^2 \\
&\qquad - \int_\Omega \left(\int_\xi^1 \frac{bP}{p} \operatorname{grad} T\, d\xi' \right) \cdot v - \int_\Omega \left[\int_\xi^1 \frac{abP}{p} \operatorname{grad}(q_1 T)\, d\xi' \right] \cdot v. \qquad (6.11)
\end{aligned}
$$

Similarly to (6.11), we get

$$
\begin{aligned}
&\frac{1}{2} \frac{d|T|_2^2}{dt} + \int_\Omega |\nabla T|^2 + \int_\Omega |T_\xi|^2 + \alpha_s |T|_{\xi=1}|_2^2 \\
&\quad \leqslant 3\varepsilon \|v\|^2 + 3\varepsilon \|T\|^2 + c|T_2|_4^8 \left(|T|_2^2 + |v|_2^2 \right) + c\left[\left(|T_{2\xi}|_2^2 + 1 \right) |\nabla T_{2\xi}|_2^2 + |T_{2\xi}|_2^2 \right] |T|_2^2 \\
&\qquad + \int_\Omega \frac{bP}{p} W(v) T + \int_\Omega \frac{abP}{p} q_1 W(v) T + \int_\Omega \frac{abP}{p} q T W(v_2), \qquad (6.12)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2} \frac{d|q|_2^2}{dt} + \int_\Omega |\nabla q|^2 + \int_\Omega |q_\xi|^2 + \beta_s |q|_{\xi=1}|_2^2 \\
&\quad \leqslant 3\varepsilon \|v\|_{H^1(\Omega)}^2 + 3\varepsilon \|q\|_{H^1(\Omega)}^2 + c\left[\left(|q_{2\xi}|_2^2 + 1 \right) |\nabla q_{2\xi}|_2^2 + |q_{2\xi}|_2^2 \right] |q|_2^2 \\
&\qquad + c|q_2|_4^8 \left(|q|_2^2 + |v|_2^2 \right). \qquad (6.13)
\end{aligned}
$$

Similarly to (5.80), we have

$$\left|\int_\Omega \frac{abP}{p} qW(v_2)T\right| = \left|\int_\Omega \int_\xi^1 \frac{abP}{p} \operatorname{grad}(qT)\, d\xi' \cdot v_2\right|$$
$$\leqslant c|v_2|_4|q|_4|\nabla T|_2 + c|v_2|_4|T|_4|\nabla q|_2$$
$$\leqslant c|v_2|_4^8\big(|q|_2^2 + |T|_2^2\big) + 2\varepsilon\|q\|^2 + 2\varepsilon\|T\|^2. \tag{6.14}$$

By integration by parts, from (6.11)–(6.14), by using (5.6) and choosing ε small enough, we obtain

$$\frac{d(|v|_2^2 + |T|_2^2 + |q|_2^2)}{dt} + \int_\Omega \big(|\nabla_{e_\theta} v|^2 + |\nabla_{e_\varphi} v|^2 + |v|^2\big) + \int_\Omega |v_\xi|^2$$
$$+ \int_\Omega |\nabla T|^2 + \int_\Omega |T_\xi|^2 + \alpha_s|T|_{\xi=1}|_2^2 + \int_\Omega |\nabla q|^2 + \int_\Omega |q_\xi|^2 + \beta_s|q|_{\xi=1}|_2^2$$
$$\leqslant c\left[|v_2|_4^8 + |T_2|_4^8 + |q_2|_4^8 + |v_{2\xi}|_2^2 + \big(|v_{2\xi}|_2^2 + 1\big)\int_\Omega \big(|\nabla_{e_\theta} v_{2\xi}|^2 + |\nabla_{e_\varphi} v_{2\xi}|^2\big)\right]|v|_2^2$$
$$+ c\big[|v_2|_4^8 + |T_2|_4^8 + |T_{2\xi}|_2^2 + \big(|T_{2\xi}|_2^2 + 1\big)|\nabla T_{2\xi}|_2^2\big]|T|_2^2$$
$$+ c\big[|v_2|_4^8 + |T_2|_4^2 + |T_2|_4^4 + |q_2|_4^8 + |q_{2\xi}|_2^2 + \big(|q_{2\xi}|_2^2 + 1\big)|\nabla q_{2\xi}|_2^2\big]|q|_2^2. \tag{6.15}$$

By the Gronwall inequality, Proposition 3.1 and (6.15), we prove Proposition 3.2. □

In fact, we have established the following result which is stronger than Proposition 3.2.

Proposition 6.1 *(The uniqueness of strong/weak solutions). Let U_1 be a weak solution to the system* (2.11)–(2.17)*. If there exists a weak solution U_2 of the system* (2.11)–(2.17) *on the interval* $[0, \mathcal{T}]$ *with the same initial conditions, such that*

$$U_2 \in L^8\big(0, \mathcal{T}; \big(L^4(\Omega)\big)^4\big), \qquad U_{2\xi} \in L^\infty\big(0, \mathcal{T}; \big(L^2(\Omega)\big)^4\big) \cap L^2\big(0, \mathcal{T}; \big(H^1(\Omega)\big)^4\big),$$

then the solutions U_1, U_2 coincide on $[0, \mathcal{T}]$.

7. The existence of universal attractors

Proof of Proposition 3.3. From (5.8), (5.63), (5.65), (5.74), (5.76), (5.85), (5.86), we know $U \in L^\infty(0, \infty; V)$ and

$$\big\|U(t)\big\| \leqslant C\big(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1\big),$$

where C is a positive constant dependent on $\|U_0\|, \|Q_1\|_1, \|Q_2\|_1$ and $0 \leqslant t \leqslant +\infty$. By Proposition 3.1 and Proposition 3.2, we can define the semigroup $\{S(t)\}_{t\geqslant 0}$ corresponding to the system (2.11)–(2.16) where $S(t): V \to V$, $S(t)U_0 = U(t)$. By (5.8), (5.63), (5.65), (5.74), (5.76), (5.85), (5.86), we prove that the corresponding semigroup $\{S(t)\}_{t\geqslant 0}$ possesses a bounded absorbing set B_ρ in V, i.e., for any $U_0 \in V$, there exists t_0 big enough such that

$$S(t)U_0 \in B_\rho, \quad \text{for any } t \geqslant t_0,$$

where $B_\rho = \{U;\ U \in V,\ \|U\| \leqslant \rho\}$ and ρ is a positive constant dependent on $\|Q_1\|_1, \|Q_2\|_1$. □

In order to prove Theorem 3.4, we need the following property about the semigroup $\{S(t)\}_{t\geqslant 0}$.

Proposition 7.1. *For every $t \geqslant 0$, the mapping $S(t)$ is weakly continuous from V to V.*

Proof. Let $\{U_n\}$ be a sequence in V such that $U_n \to U$ weakly in V. Then $\{U_n\}$ is bounded in V. By the a priori estimates in Section 5, we know that, for every $t \geqslant 0$, $\{S(t)U_n\}$ is bounded in V. So we extract a subsequence $\{S(t)U_{n_k}\}$ such that $S(t)U_{n_k} \to u$ weakly in V. Since the embedding $V \hookrightarrow L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ is compact, $U_{n_k} \to U$ strongly in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. By (6.15), we obtain that $S(t)U_{n_k} \to S(t)U$ strongly in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. Then $u = S(t)U$. Therefore, the sequence $\{S(t)U_n\}$ satisfies: $S(t)U_{n_k} \to S(t)U$ weakly in V. Proposition 7.1 is proved. □

Proof of Theorem 3.4. With Proposition 3.3 and Proposition 7.1, we know that Theorem 3.4 follows directly from Theorem I.1.1 in [32] since this is a result for a dynamical system on a general metric space. So the details of the proof for Theorem 3.4 is omitted here. □

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